

COVERING THEOREMS FOR THE CORE MODEL, AND AN APPLICATION TO STATIONARY SET REFLECTION

SEAN COX

ABSTRACT. We prove covering theorems for K , where K is the core model below the sharp for a strong cardinal, and give an application to stationary set reflection.

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1. INTRODUCTION

Jensen's covering lemma for L expressed a dichotomy: either $0^\#$ exists, or every uncountable set b of ordinals in V is covered by a set in L of the same cardinality. This had the immediate consequence that, e.g., the failure of the Singular Cardinals Hypothesis is a large cardinal property. This covering lemma cannot hold for higher core models: Prikry forcing shows that there cannot be a canonical inner model K which is resistant to set-sized forcing, has a measurable cardinal, and covers every uncountable set in its forcing extensions.

Mitchell [7] proved a kind of covering theorem for the core model below $o(\kappa) = \kappa^{++}$, and obtained a dichotomy similar to Jensen's: either K recognizes the singularity of singular V -cardinals, or else K has large cardinals.

This paper generalizes Mitchell's result to the following:

Theorem 1. (Main Theorem:) *Let K be the core model below the sharp for a strong cardinal (i.e., there is no mouse with an overlapping extender). Assume $\omega_2 < \gamma$, γ is regular in K , and $cf^V(\gamma) < |\gamma|^V$. Then γ is measurable in K .*

Moreover, if $cf^V(\gamma)$ is uncountable then $o^K(\gamma) \geq cf^V(\gamma)$ and for every $\beta < cf^V(\gamma)$, $\{\lambda < \gamma \mid o^K(\lambda) > \beta\}$ contains a closed unbounded set.

Theorem 1 generalizes Mitchell's result in several directions: it applies to a larger core model, and does not require γ to be a cardinal in V . Furthermore, in the case where $cf^V(\gamma)$ is uncountable, Theorem 1 removes the assumption that $(cf^V(\gamma))^\omega = cf^V(\gamma)$ from the main theorem of [7], answering a question from that paper. Vickers [9] had obtained results similar to the "uncountable cofinality" case of Theorem 1 at a smaller core model.¹

Theorem 1, and its proof, are especially useful in obtaining lower bounds for the consistency strength of combinatorial properties at small regular cardinals (like ω_2 and ω_3). This article gives one such application to stationary set reflection at ω_3 :

Theorem 2. *Let S_m^n denote $\omega_n \cap \text{cof}(\omega_m)$. Let $<^*$ be the relation on ${}^{\omega_3}V$ defined by: $f <^* g$ iff there is a $< \omega_2$ -closed unbounded $C \subset \omega_3$ such that $f(\alpha) < g(\alpha)$ for every $\alpha \in C$. Let h_ν denote the ν -th canonical*

¹Vickers proved the following, regarding ordinals of uncountable cofinality: assume $0^\#$ does not exist (i.e. there is no mouse with a measure of order > 0) and let K be the core model. If $\omega_2 < \gamma$, γ has uncountable cofinality, $cf^V(\gamma) < |\gamma|^V$, and γ is regular in K , then γ is measurable in K . If $\omega_2 \leq cf^V(\gamma) < |\gamma|^V$ then γ is singular in K .

function on ω_3 . Let o^K be the function defined by sending λ to its Mitchell order in K .

- (1) Assume every stationary subset of S_0^3 reflects at a point in $\text{cof}(\omega_1)$. Then $h_\nu <^* o^K$ for every $\nu < \omega_3^{+K}$.
- (2) (Simultaneous reflection) Assume every pair of stationary subsets of S_0^3 have a common reflection point in $\text{cof}(\omega_1)$. Then $o^K(\omega_3) \geq \omega_3^{+K}$.

Section 2 provides some background. Section 3 states and proves a general version of the so-called “Frequent Extensions of Embeddings Lemma.” Section 4 includes some consequences of the Frequent Extensions of Embeddings Lemma, due essentially to Mitchell. Section 5 proves Theorem 1 (and the related Theorem 47); the case where $cf(\gamma) = \omega$ is the hardest and takes up most of the section. Section 6 proves Theorem 2.

2. PRELIMINARIES

An uncountable set X will be called *weakly internally approachable* iff there is a strictly $\subset$ -increasing sequence $\langle Y_i | i < \mu \rangle$ such that $cf(\mu) > \omega$, $X = \bigcup_i Y_i$, and for every $i < \mu$: $Y_i \in X$, $Y_i \subset X$, and $|Y_i| < |X|$. If $\kappa \geq \omega_2$ is regular, the collection of weakly internally approachable X is stationary in $P_\kappa(H_\theta)$ (in fact closed under sufficiently long $\subset$ chains). If $S \subset P_\kappa(H_\theta)$ is stationary, the phrase “property P holds for almost every $X \in S$ ” means that $\{X \in S | P \text{ does not hold for } X\}$ is nonstationary. The Fodor Lemma for $P_\kappa(H_\theta)$ states that if S is stationary and F is a function on S such that $F(X) \in X$ for every $X \in S$, then there is a stationary $S' \subset S$ and a set Z such that for every $X \in S'$, $F(X) = Z$.

If A is a set of cardinality κ , a *filtration of A* is a continuous $\subset$ -increasing sequence $\langle A_\alpha | \alpha < \kappa \rangle$ such that $|A_\alpha| < \kappa$ for every $\alpha < \kappa$ and $A = \bigcup_{\alpha < \kappa} A_\alpha$. If κ is regular and uncountable then any two filtrations of A agree on a club. So if $\nu < \kappa^+$ and $\langle A_\alpha^\nu | \alpha < \kappa \rangle$ and $\langle B_\alpha^\nu | \alpha < \kappa \rangle$ are filtrations of ν , then the functions $\alpha \mapsto otp(A_\alpha^\nu)$ and $\alpha \mapsto otp(B_\alpha^\nu)$ agree on a club. The equivalence class² of $\alpha \mapsto otp(A_\alpha^\nu)$ is called the ν -th *canonical function on κ* ; in this paper h_ν will denote a representative of this class. There are also inductive definitions of h_ν ; see e.g. [3].

2.1. Fine structure and ultrapowers. All fine-structural notions used in this paper can be found in [10] (e.g. fine-structural ultrapowers). We fix some notation and state a useful lemma.

Definition 3. *If M is an acceptable J -structure, ϕ is an $\Sigma_0^{(n)}$ formula, τ is an M -cardinal with $\tau \leq \omega \rho_M^n$, and $f_1, \dots, f_k$ are good $\Sigma_1^{(n-1)}(M)$ functions with $\gamma_i := \text{dom}(f_i) \in \tau$ for each i , then $u_{f_1, \dots, f_k}^{\phi, M} := \{ \prec \xi_1, \dots, \xi_k \succ | \langle \xi_1, \dots, \xi_k \rangle \in \prod_{i \leq k} \gamma_i \text{ and } M \models \phi(f_1(\xi_1), \dots, f_k(\xi_k)) \}$; note this is a bounded subset of τ . The symbols $u_{f=g}^M$ and $u_{f \in g}^M$ will stand for the sets $\{\xi \in \text{dom}(f) \cap \text{dom}(g) | f(\xi) = g(\xi)\}$ and $\{\xi \in \text{dom}(f) \cap \text{dom}(g) | f(\xi) \in g(\xi)\}$, respectively.*

Lemma 4. *(Interpolation Lemma) Assume $\sigma : M \rightarrow_{\Sigma_0^{(n)}} M^*$ where M end extends $Q = J_\tau^A$, τ is a cardinal in M , and $\omega \rho_M^n \geq \tau$. Let $\tilde{\tau} := \sup(\sigma[\tau])$, and $Q' := M | \tilde{\tau}$. Then $M' := \text{ult}^n(M, \sigma \upharpoonright \tau)$ is wellfounded. Let $\tilde{\sigma} : M \rightarrow M'$ be the n -ultrapower map. There is also a unique $\Sigma_1^{(n-1)}$ -preserving map (if $n = 0$ this map is generally Σ_0 preserving) $\sigma' : M' \rightarrow M^*$ such that $\sigma' \upharpoonright \tau' = \text{id}$ and $\sigma' \circ \tilde{\sigma} = \sigma$. If $\tilde{\sigma}$ is n -cofinal (e.g., if $R_M^n \neq \emptyset$, or if $n = k(M, \tau)$) then σ' is $\Sigma_0^{(n)}$ preserving.*

²modulo the equivalence relation on ${}^\kappa V$ defined by “agree on a club.”

We state the following basic fact about cutpoints, which will be used in section 5:

Fact 5. *Let $\pi : M \rightarrow_F N$ and $\gamma < \pi(\kappa)$; without loss of generality, assume F is whole. Then γ is a cutpoint of F if and only if $\pi(\kappa) \cap h_N[\text{range}(\pi) \cup \gamma] = \gamma$ (i.e., γ is not a cutpoint iff $h_N[\text{range}(\pi) \cup \gamma] \cap [\gamma, \pi(\kappa)) \neq \emptyset$).*

2.2. Premice with non-overlapping extenders. This section summarizes some facts about premice for non-overlapping extenders. The definition of a bookkeeping premouse, which appears on page 252 of [10], is rather technical, and the top predicate of a bookkeeping premouse is *not* an amenable predicate (so the fine structure theory does not apply to it). If M is a bookkeeping premouse then the *expansion* of M , denoted by $\widehat{M}$ is an acceptable J-structure and is in fact a coherent structure; again see page 252 of [10] for the definition. A bookkeeping premouse and its expansion can be computed from each other. In coiterations between mice, the notion of “least disagreement” is unambiguous if we’re referring to bookkeeping premice, but is somewhat ambiguous if we’re referring to their expansions; this is why bookkeeping premice are used.

We use the following notations related to the Mitchell order (these agree with [10]):

Definition 6. *For $\kappa \in M$, $O^M(\kappa) := \{\nu \geq \kappa^{+M} \mid E_\nu^M \neq \emptyset \text{ and } cr(E_\nu^M) = \kappa\}$, $o_*^M(\kappa) = otp(O^M(\kappa))$, and $o^M(\kappa)$ is the least primitive recursively closed $\nu \geq \kappa^{+M}$ which does not index an extender with critical point κ . $o^M(\kappa)$ is called the order of κ in M , and $o_*^M(\kappa)$ is called the Mitchell order of κ in M . Note that by the coherency requirement of premice, $o_*^M(\kappa) \geq \beta$ implies that the Mitchell order (in the usual sense) of κ is at least β . If F is an extender over M with critical point κ and $ult(M, F)$ is wellfounded, we define $o(F)$ as $o^{ult(M, F)}(\kappa)$, and similarly for $o_*(F)$. If $F = E_\nu^M$ is an extender on the sequence of the premouse, then $O^M(F) := \{\eta \in [\tau, \nu) \mid E_\eta \neq \emptyset\}$ and $o_*^M(F)$ is defined as the order-type of $O^M(F)$; i.e. $O^M(F)$ is $O^{ult(\widehat{M}, F)}(\kappa)$ and $o_*^M(F)$ is $o_*^{ult(M, F)}(\kappa)$.*

Here are a few basic facts about these premice which appear in section 8.1 of [10]:

Lemma 7. (1) *If $\kappa^{+M} \leq \gamma < o^M(\kappa)$ and γ is primitive recursively closed then $cr(E_\gamma) = \kappa$;*
 (2) *If there is a (possibly partial) extender on M ’s extender sequence with critical point κ , then $o^M(\mu) < \kappa$ for every $\mu < \kappa$.*

- (3) If $M = (J_\alpha^E, E_\alpha)$ is a bookkeeping premouse and $cr(E_\alpha) = \kappa$, then $\alpha = o_{\widehat{M}}^M(\kappa)$ and $\pi(\kappa) > \alpha$, where $\pi : J_\tau^E \rightarrow_{E_\alpha} J_{\hat{\tau}}^E$ and $\tau = \kappa^{+M}$.
- (4) If F is on the extender sequence of a premouse M , then F has no cutpoints.³
- (5) The property of being a premouse without a top extender is a Q -property.
- (6) Premousehood is a Π_2 property for transitive rudimentarily closed structures. $M = (J_\alpha^E, F)$ is a premouse (i.e. the expansion of a bookkeeping premouse) iff:
 - (a) $(J_\alpha^E, \emptyset)$ is a premouse;
 - (b) M is a coherent structure;
 - (c) F is weakly amenable with respect to J_α^E . For coherent structures with top predicate F , this is equivalent to F measuring all subsets of κ in the coherent structure;
 - (d) All generators of F are $< \alpha$.
- (7) If G is a weakly amenable extender with respect to $\widehat{M}$ and $cr(G) \leq \text{height}(M)$, then $\text{ult}(\widehat{M}, G)$ is a premouse (assuming it is wellfounded).
- (8) Assume Q is an initial segment of M , $\sigma : Q \rightarrow_{\Sigma_0} Q'$ is cofinal, and $\text{ult}^n(\widehat{M}, \sigma)$ is wellfounded; let $\tilde{\sigma} : \widehat{M} \rightarrow \text{ult}^n(\widehat{M}, \sigma)$. If either:
 - $n > 0$, or
 - $n = 0$ and $cr(E_{top}^M)^{+\widehat{M}} \geq ht(Q)$
 then $\text{ult}^n(\widehat{M}, \sigma)$ is a premouse.

Let $M = (J_\alpha^E, E_\alpha)$ be a bookkeeping premouse with $E_\nu \neq \emptyset$, $\kappa = cr(E_\nu)$, and $\tau = \kappa^{+M}$. Suppose $\beta > \kappa$ is a generator for E_ν . Then β must be strictly between τ and ν , and must be a fixed point for the enumeration of p.r.c. ordinals $\geq \tau$. In other words, to be a generator β would have to be the β -th p.r.c. ordinal $\geq \tau$. Consequently, if $o_*^M(E_\nu) \leq \kappa^{+M}$ (or even a bit higher) then E_ν has only one generator, namely, its critical point. Such an extender is essentially a measure, as the ultrapower can be constructed using only $E_{\nu, \kappa}$. For this reason, we have the following convention for the rest of the article:

Convention 8. If $o_*^M(E_\nu) = \gamma$ is sufficiently small, then we will usually view E_ν as simply the measure $E_{\nu, \kappa}$, and denote it by $\mathcal{U}^M(\kappa, \gamma)$.⁴

³A cutpoint on a coherent structure like $\widehat{M}$ would yield a superstrong extender in $\widehat{M}$, which is far beyond the core model used in this paper.

⁴See Definition 6 for the meaning of o^M , o_*^M , and O^M .

Iterations of premice and the associated concepts (iteration indices, truncations, etc.) are defined in the usual way, with the slight technical difference that the domains of the iteration maps are expansions of bookkeeping premice. All iterations of premice in this paper are *linear*; i.e. iteration trees are not necessary here. Unless stated otherwise, iteration means that $*$ -ultrapowers are always used. An iteration is *simple* if there are no truncations in the iteration. Since the extender E_ν is close to $\widehat{M||\nu}$ for any premouse M , total iteration maps are Σ^* -preserving. For $k \in \omega$, a k -iteration of a premouse M is an iteration that uses k -ultrapowers until the first truncation, and then $*$ -ultrapowers at all later stages. A *degenerate* $(k-)$ iteration of M is a concatenation of ω -many simple $(k-)$ iterations which starts with M and has a truncation at each concatenation point. A premouse M is called $(k-)$ *iterable* or a *mouse* if and only if every $(k-)$ iteration of M can be continued, and there are no degenerate $(k-)$ iterations of M . An iteration is *normal* iff the iteration indices are strictly increasing, and M is called *normally* $(k-)$ *iterable* iff every normal $(k-)$ iteration of M can be continued. An iteration is *above* κ iff all the critical points of the iteration are above κ . M is *(normally) iterable above* κ iff every (normal) iteration of M above κ can be continued.

Coiterations (i.e. comparisons) of premice are defined as usual, and for mice with nonoverlapping extenders the coiteration process is linear. If M and N are coiterable then the length of the coiteration is $< \max |M|^+, |N|^+$. Furthermore, if every proper initial segment of M and N are solid (this is a Π_1 condition called *presolidity*), then at least one side of the coiteration is simple. We will see below that iterability guarantees this condition, and more.

We fix some notation regarding coiterations; for simplicity this definition will only apply to coiterations where one side is simple:

Definition 9. *Let M, N be coiterable premice, and suppose their coiteration has length $\theta+1$. Assume the M side of the coiteration is simple. The coiteration data is*

$$\langle M_i, \pi_{i,j}^M, N_i, \pi_{i,j}^N, \nu_i, \kappa_i, \tau_i, \delta_i, (N_i)^* | i \leq j \leq \theta \rangle$$

where:

- For $i < \theta$:
 - $M_i, \pi_{i,j}^M$ are the iterates and iteration maps on the M side, and similarly for the N side (if there is a truncation between stages i and j then $\pi_{i,j}^N$ is only partial).
 - ν_i is the least point of disagreement between N_i and M_i

- κ_i is the corresponding critical point of whichever side has an extender indexed at ν_i (if ν_i indexes an extender in both M_i and N_i then $cr(E_{\nu_i}^{M_i}) = cr(E_{\nu_i}^{N_i})$).
- $\tau_i = \kappa_i^{+M_i}$
- δ_i is maximal such that τ_i is a cardinal in $\widehat{N_i || \delta_i}$
- $(N_i)^* := N_i || \delta_i$; note that if $\delta_i < ht(N_i)$ then the ultimate projectum of $\widehat{(N_i)^*}$ is $\leq \kappa_i$ and $\widehat{(N_i)^*}$ is fully sound.
- For $i = \theta$: if M_θ is a proper initial segment of $N_\theta || \kappa^{+N_\theta}$ for some κ , then we define $\kappa_\theta := \kappa$, $\tau_\theta = \kappa^{+M_\theta}$, δ_θ to be maximal so that τ_θ is a cardinal in $N_\theta || \delta_\theta$, and $(N_\theta)^* := N_\theta || \delta_\theta$. Note that such a κ_θ is not literally a critical point of the coiteration.

The following is often called the *copying construction*; we state the version for k -iterations:

Definition 10. Assume $\bar{M}, M$ are premice, $\sigma : \widehat{\bar{M}} \rightarrow_{\Sigma_0^{(k)}} \widehat{M}$, and $\bar{\mathcal{I}}$ is a k -iteration of $\bar{M}$. The k -copy of $\bar{\mathcal{I}}$ via σ consists of the following (which may be vacuous; the next lemma gives sufficient conditions for this copying construction to be carried out):

- A k -iteration $\mathcal{I}$ of M of the same length as $\bar{\mathcal{I}}$; denote this length by θ and the iteration maps by $\pi_{i,j}$;
- For each $i < \theta$ a map $\sigma_i : \widehat{\bar{M}_i} \rightarrow_{\Sigma_0^{(k)}} \widehat{M_i}$ where $\sigma_0 = \sigma$, $\pi_{i,j} \circ \sigma_i = \sigma_j \circ \bar{\pi}_{i,j}$ and $\sigma_j \upharpoonright \nu_i = \sigma_i \upharpoonright \nu_i$ for every $i \leq j < \theta$. The maps σ_i are called copies of σ .
- $\sigma_i(\bar{\nu}_i) = \nu_i$ for every $i < \theta$.

Note that if $\bar{\mathcal{I}}$ is normal and its k -copy to M exists, then the copy is also normal. Also, the truncation stages of the copy are the same as the truncation stages of $\bar{\mathcal{I}}$. The following is a sufficient condition for the copying construction to be carried out; again we state the version for k -iterations:

Lemma 11. Assume $\sigma : \widehat{\bar{M}} \rightarrow_{\Sigma_0^{(k)}} \widehat{M}$, $R_{\widehat{\bar{M}}}^k \neq \emptyset$ if $k > 0$, and M is k -iterable. Then the k -copying construction can be carried out.

Proof. The proof is similar to Lemma 4.3.1 in [10]. In the current version, the assumption that $R_{\widehat{\bar{M}}}^k \neq \emptyset$ (if $k > 0$) is used to ensure that the ultrapower maps in the iterations and the copy maps remain $\Sigma_0^{(k)}$ preserving (even if the degree of the critical points of the coiteration are strictly less than k). $\square$

The copying construction (for $*$ -iterations) is used to prove the Dodd-Jensen Lemma, which states that if M is iterable, $\mathcal{I}$ is an iteration of

M resulting in M' , and there is a Σ^* -preserving map $\sigma : \widehat{M} \rightarrow \widehat{M}'$, then $\mathcal{I}$ is simple and the iteration map from $M \rightarrow M'$ is pointwise smaller than σ . One consequence of this is that a given mouse Q cannot be both a simple and a non-simple iterate of M .

There is also a version of the Dodd-Jensen Lemma for k -iterable mice which have k -very good parameters, and uses the copying construction in Definition 10:

Lemma 12. *Assume M is k -iterable, $R_{\widehat{M}}^k \neq \emptyset$ if $k > 0$, there is some $\sigma : \widehat{M} \rightarrow_{\Sigma_0^{(k)}} \widehat{M}'$, and M' is a k -iterate of M . Then there is only one k -iteration $\mathcal{I}$ from M to M' , this k -iteration is simple, and the corresponding map is pointwise $\leq \sigma$.*

The *Solidity Theorem* states that every mouse is solid. This has several important consequences, including the Condensation Lemma and the fact that at least one side of a coiteration of mice is simple. The latter leads to the canonical wellordering of mice, where $M <^* N$ means that the N side truncates in the coiteration of M vs. N ; this is equivalent to the statement that there is some mouse Q which is a nonsimple iterate of N and a simple iterate of M . The following is an often-used consequence of solidity (and the Dodd-Jensen lemma):

Lemma 13. *Assume M is iterable and $\sigma : \widehat{M} \rightarrow_{\Sigma_0^{(n)}} \widehat{M}$, and that in the “attempted” coiteration of $\bar{M}$ with $\bar{M}$, the $\bar{M}$ side is above $\omega \rho_{\widehat{M}}^{n+1}$; i.e. $\bar{M}$ and M agree below $o^{\bar{M}}(\omega \rho_{\widehat{M}}^{n+1})$. Then $\bar{M}$ and M are coiterable, the $\bar{M}$ side is simple, and $\bar{M}$ is iterable.*

We also state a version for k -iterations:

Lemma 14. *Assume M is k -iterable, $\sigma : \widehat{M} \rightarrow_{\Sigma_0^{(k)}} \widehat{M}$, and $R_{\widehat{M}}^k$ and $R_{\widehat{\bar{M}}}^k$ are both nonempty if $k > 0$. Then $\bar{M}$ and M are k -coiterable, the $\bar{M}$ side is simple, and $\bar{M}$ is k -iterable.*

Proof. Since M is k -iterable then every proper initial segment is $*$ -iterable; so the usual Solidity Theorem implies $\widehat{M}$ is presolid. Presolidity is a Π_1 property, and so $\widehat{\bar{M}}$ is presolid as well. Furthermore, the preservation degree of σ and $R_{\widehat{M}}^k \neq \emptyset$ (for $k > 0$) implies by Lemma 11 that $\bar{M}$ is k -iterable. Let Q be the last premouse in the k -coiteration of $\bar{M}$ with M . Since both $\bar{M}, M$ are presolid then at least one side of the k -coiteration is simple. The $\bar{M}$ side must be simple. If not, let $\pi_{M,Q}$ be the k -iteration map on the (simple) M side; $R_{\widehat{M}}^k \neq \emptyset$ guarantees that $\pi_{M,Q}$ is $\Sigma_0^{(k)}$ preserving, even if the degrees of the critical points used

in the iteration are strictly less than k . Then $\pi_{M,Q} \circ \sigma$ would be a $\Sigma_0^{(k)}$ map from $\widehat{M} \rightarrow Q$, but Q is also a non-simple k -iterate of $\widehat{M}$. This contradicts Lemma 12. $\square$

The following lemma appears as Lemma 8.2.2 in [10].

Lemma 15. *Assume $\sigma : \widehat{M} \rightarrow_{\Sigma_0} \widehat{N}$ where M, N are 0-iterable. If $\mu < cr(\sigma)$, then M, N agree below $o^M(\mu)$. Consequently, if these two premice are coiterable, then the M side is above $o^M(\mu)$.*

In this paper, lemmas 13/14 and 15 are often used in conjunction; Lemma 15 is used first to show that the conditions of Lemma 13 or 14 are met.

The following are versions of Lemmas 8.2.10 and 8.2.11 in [10]:

Lemma 16. *Assume M is a mouse which end-extends $Q = J_\tau^E$, κ is the largest cardinal in Q , $\widehat{M}$ is sound above κ , $cf(\tau) > \omega$, $n = degree_{\widehat{M}}(\kappa)$, and $\sigma : Q \rightarrow_{\Sigma_0} \tilde{Q}$ is cofinal. Then $P := ult^*(\widehat{M}, \sigma)$ is wellfounded. If P is a premouse then it is n -iterable. If P is not a premouse, then the “premouse associated with P ” (defined below) is m -iterable, where $\mu = cr(E_{top}^P)$ and m is the degree of μ in $\widehat{P}||\delta$.*

Although the statement of lemma 16 in [10] only mentions that $\tilde{M}$ is normally iterable above $\sigma(\kappa)$, the proof there—in conjunction with Lemma 14—really shows that $\tilde{M}$ is fully n -iterable. In this paper, Lemma 16 will be used only when protomice arise during the proof of the Frequent Extensions of Embeddings Lemma. In those cases, the “premouse replacements” for the protomice can be viewed as canonical liftings of the sort guaranteed by Lemma 16.

There is also the coarse version of the last lemma:

Lemma 17. *Assume M is a 0-iterable premouse which end-extends $Q = J_\tau^E$, $cf(\tau) > \omega$, Q has a largest cardinal κ , $o^M(\mu) < \tau$ for every $\mu < \kappa$, and $\sigma : Q \rightarrow_{\Sigma_0} \tilde{Q}$ is cofinal. Then the coarse ultrapower $ult(\widehat{M}, \sigma)$ is wellfounded and 0-iterable.*

Corollary 18. *Assume M is a mouse which is sound above κ , F is an extender over M with critical point κ , and $\tau := \kappa^{+\widehat{M}}$ has uncountable cofinality. If $ult(M|\tau, F)$ is wellfounded (recall that $M|\tau$ denotes J_τ^E , i.e. $M||\tau$ without the top predicate), then $ult^*(\widehat{M}, F)$ is a wellfounded premouse and is normally iterable above $\pi(\kappa)$, where π is the *-ultrapower map of $\widehat{M}$ by F .*

Proof. This is a consequence of Lemma 16; note $ult^*(\widehat{M}, F) = ult^*(\widehat{M}, \pi)$ where $\pi : M|\tau \rightarrow_F ult(M|\tau, F)$. $\square$

Corollary 19. *Assume M is a 0-iterable bookkeeping premouse, κ is a cardinal in $\widehat{M}$, $\tau := \kappa^{+\widehat{M}}$ has uncountable cofinality, and $o^M(\mu) < \tau$ for every $\mu < \kappa$. Let F be an extender over M with critical point κ . If $ult(M|_\tau, F)$ is wellfounded, then $ult(\widehat{M}, F)$ is a wellfounded 0-iterable premouse.*

Proof. This is a consequence of Lemma 17; note $ult(\widehat{M}, F) = ult(\widehat{M}, \pi)$ where $\pi : M|_\tau \rightarrow_F ult(M|_\tau, F)$. $\square$

The following lemma is a useful condition which guarantees the uniqueness of an extender that can appear on a premouse sequence.

Lemma 20. *Assume $M = (J_\alpha^E, F_M)$ and $N = (J_\alpha^E, F_N)$ are 0-iterable bookkeeping premice, $\kappa := cr(F_M) = cr(F_N)$, and there is some $\tau \in (\kappa, \alpha]$ of uncountable cofinality which is a successor cardinal in both $\widehat{M}, \widehat{N}$. Then $M = N$ (i.e. $F_M = F_N$).*

Proof.

Claim 21. *$\widehat{M}, \widehat{N}$ have the same successor of α and agree below this successor.*

We show that $\alpha^{+\widehat{M}} \leq \alpha^{+\widehat{N}}$ and that they agree below $\alpha^{+\widehat{M}}$; a symmetric argument shows the reverse.

Consider the ultrapower map $\pi_N : J_\tau^E \rightarrow_{F_N} J_{\tau'_N}^{E^{N'}}$; note that since τ is a successor cardinal then $J_\tau^E \models ZFC^-$, so $J_{\tau'_N}^{E^{N'}}$ is an initial segment of $ult(J_\alpha^E, F_N)$, which in turn is the domain of the expansion $\widehat{N}$. Since M end-extends J_τ^E (note we do *not* require that τ is the successor of κ ; τ might be larger than κ^{+M}) and $\alpha = o^{\widehat{M}}(\kappa)$ by Lemma 7, then Lemma 17 guarantees the coarse lifting of π_N to $\widehat{M}$ is wellfounded and 0-iterable. Let $\pi : \widehat{M} \rightarrow \widehat{M}'$ be this coarse lifting (note it is just the ultrapower map of $\widehat{M}$ by F_N). So $\widehat{M}'$ agrees with $\widehat{N}$ below the common cardinal $\pi(\tau)$. In particular, they have the same α^+ and agree below it. Furthermore, by coherency of F_N we have $E^{M'} \restriction \alpha = E \restriction \alpha$ and $E_\alpha^{M'} = \emptyset$. So at the first stage of the 0-coiteration of M with M' , the M side applies F_M and the M' side does nothing. Let $\widehat{M}_1 = ult(\widehat{M}, F_M)$ denote the first iterate on the M side; note that the ultrapower used to create M_1 extends the coherency map $c_M : J_{\kappa+M}^E \rightarrow_{F_M} |M|$ and that $c_M(\kappa) > \alpha$. So M_1 and M have the same α^+ and agree below it.

The remainder of the coiteration is strictly above α , and by the coarse version of the Dodd-Jensen Lemma the M_1 side is simple. So $\alpha^{+\widehat{M}_1} \leq \alpha^{+\widehat{M}'}$ and $\widehat{M}_1$ agrees with $\widehat{M}'$ below $\alpha^{+\widehat{M}}$. And $\widehat{M}'$ agrees with $\widehat{N}$ on the same part.

Claim 22. $F_M = F_N$.

Let λ be the common cardinal predecessor of τ in M, N . For a large regular θ , fix some countable elementary substructure $X \prec (H_\theta, \in, M, N, \tau, \lambda, \alpha, \dots)$ and let $\sigma : \bar{H} \rightarrow H_\theta$ be the inverse of the Mostowski collapsing map. Let $(\bar{M}, \bar{N}, \bar{\tau}, \bar{\lambda}, \bar{\alpha}, \dots) = \sigma^{-1}(M, N, \tau, \lambda, \alpha, \dots)$. We will show that $\bar{F}_M = \bar{F}_N$.

Let $\tilde{\tau} = \sup(\sigma[\bar{\tau}])$; this is $< \tau$ because $cf(\tau)$ is uncountable. Using the coarse interpolation lemma, construct interpolates $\tilde{M}, \tilde{N}$ and the following maps:

- $\tilde{\sigma}_M : \widehat{\tilde{M}} \rightarrow \widehat{\tilde{M}}$ and $\tilde{\sigma}_N : \widehat{\tilde{N}} \rightarrow \widehat{\tilde{N}}$ which extend $\sigma \upharpoonright \widehat{\tilde{M}}$ and $\sigma \upharpoonright \widehat{\tilde{N}}$, respectively;
- $\sigma'_M : \widehat{\tilde{M}} \rightarrow \widehat{\tilde{M}}$ and $\sigma'_N : \widehat{\tilde{N}} \rightarrow \widehat{\tilde{N}}$ such that the critical point of both of these maps is $\tilde{\tau}$.

By Lemma (15) and the fact that $\sigma'_M \upharpoonright \tilde{\tau} = id$, then $\tilde{M}$ is an initial segment of M . In fact $\tilde{M}$ is a proper initial segment of $Q := M \upharpoonright \tau$, since $\tilde{\tau} = \lambda^{+\widehat{\tilde{M}}} < \tau = \lambda^{+\widehat{M}}$. Similarly, $\tilde{N}$ is a proper initial segment of Q .

By claim (21) and the elementarity of σ , $\widehat{\tilde{M}}$ and $\widehat{\tilde{N}}$ have the same successor of $\bar{\alpha}$ and agree below this successor. For the rest of this proof let $\bar{\alpha}^+$ denote this common successor, and let $\bar{E} := E^{\tilde{M}} \upharpoonright \bar{\alpha}^+ = E^{\tilde{N}} \upharpoonright \bar{\alpha}^+$. Now $\tilde{\sigma}_M \upharpoonright \bar{\alpha}^+ = \tilde{\sigma}_N \upharpoonright \bar{\alpha}^+$, since these restrictions are both just the coarse lifting of $\sigma \upharpoonright \bar{\tau}$ to the domain $J_{\bar{\alpha}^+}^{\bar{E}}$. Let $\tilde{\alpha}$ denote the ordinal $\tilde{\sigma}_M(\bar{\alpha}) = \tilde{\sigma}_N(\bar{\alpha})$. Then $E_{\tilde{\alpha}}^{\tilde{M}} = E_{\tilde{\alpha}}^{\tilde{N}} = E_{\tilde{\alpha}}^{\bar{E}}$, since $\tilde{M}$ and $\tilde{N}$ are both initial segments of Q . This implies that $\bar{F}_M = \bar{F}_N$, which finishes the proof. $\square$

Corollary 23. *Assume M, N are 0-iterable premice, μ is an element of both M, N , and there is some τ such that $\mu < \tau \leq \min(o^M(\mu), o^N(\mu))$, $cf(\tau)$ is uncountable, and τ is a successor cardinal in both $\widehat{M}, \widehat{N}$. Then M, N agree below $\min(o^M(\mu), o^N(\mu))$.*

Proof. Suppose β indexes an extender on μ in both premice and they agree below β . Then the pair $M \upharpoonright \beta$ and $N \upharpoonright \beta$ satisfies the hypothesis of the previous lemma, so $E_\beta^M = E_\beta^N$. $\square$

Many arguments involve an iteration of limit length where a “critical point is repeated cofinally often” in the iteration and there has been a truncation. The following lemma shows how to identify such points from the last iterate, and is essential to the proofs in section 5.

Lemma 24. *Let $\langle N_i, \pi_{i,j}, \kappa_i, \nu_i \mid i \leq j < \theta \rangle$ be a normal iteration of a mouse N_0 such that θ is a limit ordinal and there is a truncation at*

some stage $< \theta$; let $\hat{i}$ be the largest truncation stage. Assume there is a cofinal $I \subset \theta$ such that $\pi_{i,j}(\kappa_i) = \kappa_j$ for every $i < j$ which are both in I . Let $\tilde{\kappa} := \sup\{\kappa_i \mid i < \theta\}$ and let n be the degree of $\tilde{\kappa}$ in $\widehat{N}_\theta$ (i.e. n is the maximal natural number so that $\tilde{\kappa}$ is strictly less than the n -th projectum).

Then $D := \{\kappa_i \mid \hat{i} < i < \theta \text{ and } \pi_{i,\theta}(\kappa_i) = \tilde{\kappa}\}$ is a closed unbounded subset of $\tilde{\kappa}^5$ and for every $\beta \in [\min(D), \tilde{\kappa}]$: $\beta \in D$ iff $\tilde{h}[\beta]$ does not intersect $[\beta, \tilde{\kappa})$, where $\tilde{h} := \tilde{h}_{\widehat{N}_\theta}^{n+1}(-, p_{\widehat{N}_\theta})$.

Proof. That D is a closed and unbounded subset of $\tilde{\kappa}$ follows from the existence of the cofinal set I and basic properties of direct limits. The rest of the proof is about the characterization of D using the fine structure of $\widehat{N}_\theta$.

For the forward direction, assume $\beta \in D$. Then it is a critical point on the thread to $\tilde{\kappa}$; say $\beta = \kappa_i$. So for any $\xi < \beta$, $\tilde{h}(\xi) = \tilde{h}(\pi_{i,\theta}(\xi))$ is in the range of $\pi_{i,\theta}$ and thus can't be in the interval $[\kappa_i, \tilde{\kappa} = \pi_{i,\theta}(\kappa_i))$.

Since $\hat{i}$ is a truncation stage, then for every $i \in (\hat{i}, \theta]$ the ultimate projectum of $\widehat{N}_i$ is $\leq \kappa_i$ and $\widehat{N}_i$ is sound above κ_i (and the same is true of the truncated version of $\widehat{N}_i$). This is a straightforward proof by induction on the stages $\geq \hat{i}$; see e.g. the proof of Lemma 6.6.4 in [10].

From now on, assume $\beta \notin D$ and $\beta > \min(D)$. For such β , let i be the minimal such that $\beta \in \text{range}(\pi_{i,\theta})$ and $i \geq i_0$, where $i_0 > \hat{i}$ is the stage such that $\kappa_{i_0} = \min(D)$. Since we assume $\beta \geq \min(D)$, then in fact $i > i_0$ and is thus a successor stage; in fact, i is the first stage where a thread to β appears. To see that $i > i_0$, just note that $\beta \in (\kappa_{i_0}, \tilde{\kappa})$ yet $\text{range}(\pi_{i_0,\theta})$ does not intersect $[\kappa_{i_0}, \tilde{\kappa})$ (since $\tilde{\kappa} = \pi_{i_0,\theta}(\kappa_{i_0})$). So $i = j + 1$ for some stage j . Also, we may WLOG assume the degrees of the critical points past stage i_0 are fixed at n (in fact all points on the thread D are strictly between the $(n+1)$ -st and n -th projecta) and the iteration maps restricted to the n -th reducts agree with the coarse ultrapower on those reducts. Let $\beta_i = (\pi)_{i,\theta}^{-1}(\beta)$. If $\beta_i < \pi_{j,j+1}(\kappa_j)$, then since β_i is not a cutpoint of E_{ν_j} (mice below 0-pistol do not have cutpoints; see Lemma 7), Fact 5 implies that there is some $\xi < \beta_i$ and $f \in H_{\widehat{N}_j}^n$ such that $\pi_{j,j+1}(f)(\xi) \in [\beta_i, \pi_{j,j+1}(\kappa_j))$. Since $\widehat{N}_j$ is sound above κ_j then $\pi_{j,j+1}(f)(\xi)$ is $\Sigma_1^{(n)}(\widehat{N}_{j+1})$ definable using a parameter $< \kappa_j$ and $p_{\widehat{N}_{j+1}}$. Applying $\pi_{j+1,\theta}$ to $\pi_{j,j+1}(f)(\xi)$ yields an element of $\tilde{h}[\beta] \cap [\beta, \tilde{\kappa})$ and finishes the case.

⁵We do *not* assume $\tilde{\kappa}$ has uncountable cofinality.

Similarly, if $\beta_i \geq \pi_{j,j+1}(\kappa_j)$ (or simply above the strict supremum s_j of E_{ν_j} 's generators) then there is some $\xi < s_j \leq \beta_i$ and good $\Sigma_1^{(n-1)}(\widehat{N_j})$ function f in parameter r such that $\beta_i = \pi_{j,j+1}(f)(\xi)$. Again, since $\widehat{N_j}$ is sound above κ_j , $\pi_{j,j+1}(f)(\xi)$ is $\Sigma_1^{(n)}(\widehat{N_{j+1}})$ definable from ξ, p_{j+1} , and some parameter $< \kappa_j$. Then apply $\pi_{j+1,\theta}$ to $\pi_{j,j+1}(f)(\xi)$ to yield an element of $\tilde{h}[\beta] \cap [\beta, \theta)$. $\square$

2.3. Protomice. Suppose Q is an initial segment of a premouse M , and $\sigma : Q \rightarrow Q'$ is Σ_0 and cofinal. Assume $\omega\rho_{\widehat{M}}^n \geq \gamma := ht(Q)$, and that $\tilde{\sigma} : \widehat{M} \rightarrow P$ is the canonical n -lifting of σ to $\widehat{M}$. If $n > 0$ then $\tilde{\sigma}$ is $\Sigma_2^{(n-1)}$ preserving; since premousehood is a Π_2 property, this implies that P is a premouse (i.e. the expansion of a bookkeeping premouse).

Now assume $n = 0$ and $\hat{F}$ is the top extender of $\widehat{M}$ with critical point μ . If $ht(Q)$ is a successor cardinal in $\widehat{M}$ with cardinal predecessor κ , then the coarse lifting $\tilde{\sigma}$ is continuous on $\widehat{M}$ -cofinalities $> \kappa$; so if $\mu \geq \kappa$ then $\mu^{+\widehat{M}}$ is mapped continuously by $\tilde{\sigma}$, and this implies that again P is a premouse. Similarly, if γ is limit and $\mu \geq \gamma$ then P is a premouse.

The remaining case is when $n = 0$ and $\mu^{+\widehat{M}} < \gamma$. Since $\tilde{\sigma}$ is Σ_0 and cofinal, P is still a coherent structure, but its top extender $\tilde{F}$ may not be weakly amenable. If this happens, P is called a *protomouse*. Let $c_P : |P| \rightarrow_{\tilde{F}} |P|$ witness that P is a coherent structure. Since $\tilde{F}$ is not weakly amenable, $\tilde{\tau} < \tau := \mu^{+Q'}$ (i.e. $\tilde{F}$ only measures subsets of μ which are in $P|\tilde{\tau}$). Let δ be maximal such that $\tilde{\tau}$ is a cardinal in $\widehat{P||\delta}$, and let $m := \text{degree}_{\widehat{P||\delta}}(\mu)$. Assume $\text{ult}^*(\widehat{P||\delta}, \tilde{F})$ is wellfounded and let π be the ultrapower map; note that π is also the canonical lifting of c_P to $\widehat{P||\delta}$. Since $\widehat{P||\delta}$ is sound above μ (it is a proper initial segment of the premouse $|P|$), then π is $\Sigma_0^{(m)}$ preserving and m -cofinal (and $\Sigma_2^{(k)}$ -preserving for $k < m$). Then $\text{ult}^*(\widehat{P||\delta}, \tilde{F})$ is a premouse, since $\pi \upharpoonright \mu = id$ and the successor of μ is mapped continuously by π (due to the m -cofinality of π).

Definition 25. Let P be a protomouse with top extender $\tilde{F}$, and δ be maximal such that $\tilde{F}$ measures all subsets of $cr(\tilde{F})$ which are in $\widehat{P||\delta}$. If $\text{ult}^*(\widehat{P||\delta}, \tilde{F})$ is wellfounded, it is called the premouse associated with the protomouse P .

Finally, Lemma 16 implies that if $cf(\tilde{\tau}) > \omega$ then R exists (i.e. is wellfounded) and is fully m -iterable. The following summarizes all we will need to use about premice associated with protomice:

Lemma 26. • $ht(P) = \lambda^{+\hat{R}}$ where λ is the largest cardinal in the coherent structure P

- Let s be the sup of generators of $\tilde{F}$. Then $m = \text{degree}_{\hat{R}}(s)$ and $\hat{R}$ is sound above s
- The top (non weakly-amenable) extender of P can be constructed from R , since the range of the ultrapower map $\pi : \widehat{Q||\delta} \rightarrow_{\hat{F}} \hat{R}$ is $\tilde{h}_{\hat{R}}^{m+1}[\pi(\kappa) \cup \{r\}]$, where $r = \pi(p_{\widehat{P||\delta}})$.
- If $cf(\tilde{\tau}) > \omega$ then R is wellfounded and fully m -iterable.

2.4. 0-pistol and the core model for non-overlapping extenders. The premice described in section 2.2 do not have overlapping extenders; i.e. the length of one extender is never $\geq$ the critical point of another extender. An object in the universe which resembles the premice described in the definition above, but has overlapping extenders, is called a *p-mouse*. It can be shown that ω is the first projectum of any p-mouse, and that the transitive collapse of $h_M[\omega]$ does not depend on which p-mouse M is used. This transitive collapse is called *0-pistol*.

In the absence of 0-pistol, the theory of mice can be developed using linear iterations (i.e. without iteration trees). A *weasel* is a proper class mouse. A weasel W is *universal* iff whenever M is a set-size premouse which is coiterable with W , then the coiteration has length $< On$; if this holds then the M side will always be simple.

In the absence of 0-pistol, the “core model for non-overlapping extenders” can be constructed. In this paper, K refers to this core model. This core model is capable of having a single strong cardinal. Below we list a few key features of K which will be used in this paper; see [10] for more details on 0-pistol and the construction and properties of K .

Definition 27. Let W be a universal weasel and F a (κ, ν) extender. Then F is W -correct iff F is weakly amenable with respect to W , $\text{ult}(W, F)$ is wellfounded, $\nu = o^{\text{ult}(W, F)}(\kappa)$ and $E^W \restriction \nu = E^{\text{ult}(W, F)} \restriction \nu$.

Lemma 28. (Essential facts about K): Assume 0-pistol does not exist, and let K be the core model. Then:

- (1) K is a universal weasel.
- (2) If G is generic for some $\mathbb{P} \in V$ then 0-pistol does not exist in $V[G]$ and $K^{V[G]} = K$;
- (3) If F is a K -correct extender then $F \in K$.
- (4) Assume $j : K \rightarrow K'$ is elementary with critical point κ , and let F be the $(\kappa, j(\kappa))$ -extender derived from j . Then $\nu := o^{K'}(\kappa) < j(\kappa)$, all generators of F are $< \nu$, and $F|_\nu = E_\nu^K$.

Furthermore, K' is a normal iterate of K and j is the corresponding iteration map.

Corollary 29. *Assume 0-pistol does not exist, F is an extender with respect to K , and $\text{ult}(K, F)$ is wellfounded. Then $F \in K$.*

A special case is when $\mathcal{W}$ is an ultrafilter on $P^K(\kappa)$ and $\mathcal{W}$ is normal with respect to K (i.e. for every $f : \kappa \rightarrow \mathcal{W}$ in K , $\Delta f \in \mathcal{W}$). If $\text{ult}(K, \mathcal{W})$ is wellfounded and $j : K \rightarrow_{\mathcal{W}} K'$, then $\mathcal{W} \in K$ and generates the extender $E_{o_{K'}(\kappa)}^K$.

Lemma 30. *Assume $\mathcal{W}$ is an ultrafilter on $P^K(\kappa)$ which is κ -complete and weakly amenable with respect to K . Assume $\tau := \kappa^{+K}$ has uncountable cofinality. Then:*

- (1) $\text{ult}(K|\tau, \mathcal{W})$ is wellfounded
- (2) If $o^K(\mu) < \tau$ for every $\mu < \kappa$, then also $\text{ult}(K, \mathcal{W})$ is wellfounded.

Proof. Assume (1) fails, and let $\{f_n | n \in \omega\}$ witness the illfoundedness of $\text{ult}(K|\tau, \mathcal{W})$. Since $\text{cf}(\tau) > \omega$, there is some $\tilde{\tau} < \tau$ such that $\{f_n | n \in \omega\} \subseteq K|\tilde{\tau}$. Since $\mathcal{W}$ is weakly amenable with respect to K , then $\tilde{\mathcal{W}} := \mathcal{W} \cap K|\tilde{\tau}$ is an element of K , and in fact of the transitive ZFC^- model $K|\tau$ (by acceptability). Then $K|\tau$ sees that $\text{ult}(K|\tilde{\tau}, \tilde{\mathcal{W}})$ is illfounded; but then $\mathcal{W}$ is not complete with respect to ω -sequences in K .

Item (2) follows from (1) and Corollary 19. □

3. FREQUENT EXTENSIONS OF EMBEDDINGS LEMMA

This section is devoted to the proof of the Frequent Extensions of Embeddings Lemma for mice without overlapping extenders. The version presented here is more general than the versions in [10] and [5], though the main outline of the proof (originally due to Jensen) is the same. The generalizations include:

- (G1): The mouse Q in the hypothesis of Lemma 31 does not necessarily have a largest cardinal;
- (G2): We do not assume the mice M_X in the hypothesis of Lemma 31 are sound above γ_X .

The part of the proof most affected by the generalizations (G1) and (G2) is the comparison of the premice which appear, e.g., in Claim 35. Lemma 20 and its corollary 23 are used in the comparison of those premice.

The generalization (G1) is important for the applications in this paper, e.g. the proof of Theorem 45.

Lemma 31. (*Frequent Extensions of Embeddings Lemma*) *Assume:*

- $\kappa \ll \theta$ are regular cardinals and $\kappa \geq \omega_2$
- γ is an ordinal $\geq \kappa$
- $Q = J_\gamma^E$ is a mouse
- If there is some $\mu < \gamma$ such that $o^Q(\mu) = \gamma$ (i.e. Q has unboundedly many extenders with critical point μ), we consider two cases:
CASE (A): $\gamma > \mu^{+Q}$
CASE (B): μ is the largest cardinal in Q
If CASE (A) holds, we require:

- (1) There is some $\lambda \in [\mu, \gamma)$ such that $cf(\lambda^{+Q}) > \omega$

If CASE (B) holds, we make no additional requirements.

- S is a stationary set of weakly internally approachable $X \in P_\kappa(H_\theta)$ such that $(Q, \kappa, \gamma, \dots) \in X$, and if (CASE (A)) holds we assume $\lambda \in X$.
- For every $X \in S$:
 - $\pi_X : H_X \rightarrow H_\theta$ is the inverse of the Mostowski collapsing map of X .
 - $Q_X = \pi_X^{-1}(Q)$
 - $\sigma_X := \pi_X \upharpoonright Q_X$
 - $\gamma_X = \sigma_X^{-1}(\gamma) = ht(Q_X)$
 - $\tilde{\gamma}_X = \sup(\sigma_X[\gamma_X])$
 - For every $X \subseteq Y$ in S , $\sigma_{XY} = \sigma_Y^{-1} \circ \sigma_X$

- M_X is a premouse which end-extends Q_X and γ_X is regular in $\widehat{M_X}$
- There is an $n_X \in \omega$ such that $\gamma_X \leq \omega \rho_{\widehat{M_X}}^{n_X}$, $R_{\widehat{M_X}}^{n_X} \neq \emptyset$ if $n_X > 0$, γ is not $\Sigma_1^{(n_X-1)}(\widehat{M_X})$ -singularized, and M_X is fully n_X -iterable.

Then for almost every $X \in S$:

- $\text{ult}^{n_X}(\widehat{M_X}, \sigma_X)$ is wellfounded. Let F_X denote the top predicate of N_X .
- If N_X is a premouse—i.e. F_X is total on N_X —then N_X is fully n_X -iterable.
- If N_X is not a premouse, then letting δ_X be maximal such that F_X is a total extender on $N_X \parallel \delta$, then $\text{ult}^*(\widehat{N_X \parallel \delta_X}, F_X)$ is fully m_X -iterable, where m_X is the degree of μ_X in $N_X \parallel \delta_X$.

Note that if $\gamma = \kappa$, then we trivially have the conclusion, since in that case $\sigma_X \upharpoonright Q_X$ is the identity map for almost every $X \in S$ (i.e. $\tilde{\gamma}_X = \gamma_X$). So from now on, assume $\kappa < \gamma$. We concentrate mostly on (CASE (A)), since this is the difficult case. Some parts of the proof will also vary depending on whether or not $\mu < \kappa$.

For almost every $X \in S$, we have $X \prec (H_\theta, \in, \mu, \kappa, \dots)$; without loss of generality assume this holds for every $X \in S$, and let $(\mu_X, \lambda_X, \dots) = \pi_X^{-1}(\mu, \lambda, \dots)$. By the κ -completeness of the nonstationary ideal on $P_\kappa(H_\theta)$, there is some $n \in \omega$ such that $n_X = n$ for stationarily many $X \in S$; without loss of generality suppose $n_X = n$ for every $X \in S$. Furthermore, since every $X \in S$ is weakly internally approachable, then $X \cap \text{Ord}$ is ω -closed. To see this, just notice that whenever $a \subset X$ is a countable set of ordinals, weak internal approachability of X implies there is some $b \in X$ such that $a \subseteq b \subset X$. Then $\sup(b) \in X$, so a cannot be unbounded in $X \cap \text{Ord}$. Let β be the least element of X which is $\geq \sup(a)$. Then $\sup(\beta \cap b)$ is an element of X , and it is clear that $\sup(\beta \cap b) = \sup(a)$.

The ω -closure of $X \cap \text{Ord}$ implies that:

$$(2) \quad \pi_X \text{ is continuous on } \text{cof}(\omega)$$

If (CASE (A)) of Lemma 31 holds, then (2) guarantees:

$$(3) \quad \text{cf}(\lambda_X^{+Q_X}) > \omega$$

3.1. Wellfoundedness of $\widehat{ult^n}(\widehat{M_X}, \sigma_X)$. We first prove that for almost every $X \in S$, $\widehat{ult^n}(\widehat{M_X}, \sigma_X)$ is wellfounded; note that this ultrapower uses good $\Sigma_1^{(n-1)}(\widehat{M_X})$ functions. Suppose this fails; WLOG assume it fails for every $X \in S$. For each $X \in S$, let $\langle (f_i^X, r_i^X, \alpha_i^X) \mid i \in \omega \rangle$ witness the illfoundedness of the canonical lifting, where:

- f_i^X is a good $\Sigma_1^{(n-1)}(\widehat{M_X})$ function in parameter r_i^X , and $\text{dom}(f_i^X) < \gamma_X$. If CASE (B) holds—i.e. μ is the largest cardinal in Q —then we can without loss of generality assume that $\text{dom}(f_i^X) = \mu_X$ for every $X \in S$ and $i \in \omega$
- $\alpha_i^X < \sigma(\text{dom}(f_i^X))$
- For every $i \in \omega$, $\langle \alpha_{i+1}^X, \alpha_i^X \rangle \in \sigma(u_{i+1,i}^{M_X})$, where $u_{i+1,i}^{M_X} := u_{f_{i+1}^X \in f_i^X}^{\widehat{M_X}}$ (see Definition 3).

Claim 32. *Without loss of generality, we may assume that the diagram is minimal; i.e. that for every $X \in S$ there is no $\zeta_X < \text{ht}(Q_X)$, $m_X \in \omega$, and premouse R_X such that:*

- ζ_X is regular in $\widehat{R_X}$, $\zeta_X \leq \omega \rho_{\widehat{R_X}}^{m_X}$ and ζ_X is not $\Sigma_1^{(m_X-1)}(\widehat{R_X})$ -singularized, and $R_{\widehat{R_X}}^{m_X} \neq \emptyset$ if $m_X > 0$.
- R_X is m_X -iterable.
- $\widehat{ult^{m_X}}(\widehat{R_X}, \sigma_X \upharpoonright \zeta_X)$ is illfounded

Proof. If this does not hold for every $X \in S$, let $T \subseteq S$ be the collection of X where it fails. If T is nonstationary, just replace S with $S - T$. If T is stationary, for every $X \in T$ select ζ_X to be the minimal object witnessing the first bullet above. Using the Fodor Lemma with the regressive map $X \mapsto \sigma_X(\zeta_X)$ on T , there is some stationary $T' \subseteq T$ such that for every pair $X, Y \in T'$ where $X \subseteq Y$, then $\sigma_{XY}(\zeta_X) = \zeta_Y$. Then replace S with T' and then for every $X \in T'$ replace the σ_X with $\sigma_X \upharpoonright \zeta_X$. Then the resulting system is minimal. $\square$

For each $X \in S$ let $A_X \prec (H_\theta, \in)$ be countable such that $M_X \in A_X$ and $\{r_i^X \mid i \in \omega\} \subset A_X$. If $n > 0$, recall we assume that $R_{\widehat{M_X}}^n \neq \emptyset$; in this case fix some $q_X \in R_{\widehat{M_X}}^n$. Let $\sigma_X^* : H_X^* \rightarrow A_X \prec H_\theta$ be the inverse of the collapsing map, and $(M_X^*, \gamma_X^*, Q_X^*, f_i^{*X}, r_i^{*X}, q_X^*, \dots) = (\sigma_X^*)^{-1}(M_X, \gamma_X, Q_X, f_i^X, r_i^X, q_X, \dots)$. By elementarity of σ_X^* and our assumptions on $\widehat{M_X}$:

- M_X^* is n -iterable
 - $\gamma_X^* \leq \omega \rho_{\widehat{M_X^*}}^n$ and γ_X^* is not $\Sigma_1^{(n-1)}(\widehat{M_X^*})$ -singularized
 - $R_{\widehat{M_X^*}}^n \neq \emptyset$ if $n > 0$
- (4)

Claim 33. *For almost every $X \in S$, σ_X^* maps γ_X^* cofinally to its image, and so γ_X has countable cofinality.*

Proof. Suppose this fails at stationarily many $X \in S$. Let $\tilde{\gamma}_X^* < \gamma_X$ be the cofinal image of γ_X^* under σ_X^* . By the interpolation lemma, $\tilde{P}_X := \text{ult}^n(\widehat{M}_X^*, \sigma_X^* \upharpoonright \gamma_X^*)$ is wellfounded; let $\tilde{\sigma}_X^* : \widehat{M}_X^* \rightarrow_{\Sigma_0^{(n)}} \tilde{P}_X$ be the n -lifting and $\sigma'_X : \tilde{P}_X \rightarrow \widehat{M}_X$ be the map from the Interpolation Lemma. By (4) and the facts from the preliminaries section, $\tilde{\sigma}_X^*$ is n -cofinal, $\tilde{\sigma}_X^*$ maps γ_X^* continuously (i.e. to $\tilde{\gamma}_X^*$), and σ'_X is $\Sigma_0^{(n)}$ preserving. Also, if $n > 0$ then $\tilde{\sigma}_X^*(q_X^*) \in R_{\tilde{P}_X}^n$,

For $i \in \omega$, let $\tilde{f}_i^X := \tilde{\sigma}_X^*(f_i^{*X})$. Since $\sigma'_X \upharpoonright \tilde{\gamma}_X^* = \text{id}$ and is $\Sigma_0^{(n)}$ preserving, for every $i \in \omega$ we have $u_{\tilde{f}_{i+1}^X \in \tilde{f}_i^X}^{\tilde{P}_X} = u_{f_{i+1}^X \in f_i^X}^{\widehat{M}_X}$. In particular, $\text{ult}^n(\tilde{P}_X, \sigma_X \upharpoonright \tilde{\gamma}_X^*)$ is not wellfounded. We will now obtain a contradiction by showing that either $\tilde{P}_X$, or a suitable replacement for it, satisfies the properties of the R_X from Claim (32). This will contradict Claim 32, since $\tilde{\gamma}_X^* < \gamma_X$.

If $\tilde{P}_X$ is a premouse—i.e. its top predicate is weakly amenable—then Lemma 14 implies that $\tilde{P}_X$ is n -iterable. Thus the system $\langle (\tilde{P}_X, \sigma_X \upharpoonright \tilde{\gamma}_X^*) | X \in S \rangle$ constitutes a counterexample to the Frequent Extensions of Embeddings Lemma, contradicting Lemma (32).

Now assume that $\tilde{P}_X$ is not a premouse—i.e. it has a top predicate that is not weakly amenable. We will show that the premouse R_X associated with $\tilde{P}_X$ (see Definition 25) is wellfounded for each X , and that this system of R_X contradicts Lemma 32 (since $\tilde{\gamma}_X^* < \gamma_X$). Since this situation can only occur if $n = 0$, then WLOG we can assume:

(5) $\tilde{f}_i^X$ is an element of $\tilde{P}_X$ for every $i \in \omega$

Let $\pi : \widehat{M}_X \rightarrow_{E_{top}^{M_X}}^n \widehat{M}'_X$, δ_X be maximal such that $\tilde{F}_X := E_{top}^{\tilde{P}_X}$ is total in $\widehat{\tilde{P}_X} || \delta_X = \widehat{Q_X} || \delta_X$. Let $\widehat{R}'_X := \pi(\widehat{Q_X} || \delta_X)$; note R' is a proper initial segment of M'_X and so is fully iterable. Let $m = \text{degree}_{\widehat{Q_X} || \delta_X}(\mu_X)$. Define the map $j : \text{ult}^*(\widehat{Q_X} || \delta_X, \tilde{F}_X) \rightarrow \widehat{R}'_X$ by $[\eta, f]_{\tilde{F}_X}^{\widehat{Q_X} || \delta_X} \mapsto \pi(f)(\sigma'_X(\eta))$, where f is a good $\Sigma_1^{(m-1)}(\widehat{Q_X} || \delta_X)$ function with domain μ_X and $\eta < lh(\tilde{F})$. It is straightforward to check that j is well-defined and Σ_0 preserving, so the premouse R_X associated with $\tilde{P}_X$ is wellfounded. In fact, j is $\Sigma_0^{(m)}$ preserving, which implies by Lemma 14 that R_X is fully m -iterable. This yields a contradiction to Claim 32, since (5) implies that the functions $\tilde{f}_i^X$ are elements of $\widehat{R}_X$ and so $\text{ult}(\widehat{R}_X, \sigma_X \upharpoonright \tilde{\gamma}_X^*)$ is illfounded. $\square$

Since each $X \in S$ is weakly internally approachable, there is some Z_X which is an element and subset of X such that $\sigma_X \circ \sigma_X^*[Q_X^*] \subseteq Z_X$. By the Fodor Lemma for $P_\kappa(H_\theta)$ we may assume WLOG that Z_X is the same for every $X \in S$. Fix some $X_0 \in S$; then for every $X \in S$ such that $X_0 \subseteq X$, $\text{rng}(\sigma_X^*) \subseteq \text{rng}(\sigma_{X_0, X})$. From now on, Q_{X_0} will be denoted Q_0 , and WLOG we assume that every $X \in S$ is a superset of X_0 . For each $X \in S$, $\sigma_{0, X}^{-1} \circ \sigma_X^*$ is a cofinal map from Q_X^* to Q_0 . Using an interpolation-like argument and (4):

Lemma 34. $\bullet \text{ } \text{ult}^n(\widehat{M_X^*}, \sigma_{0, X}^{-1} \circ \sigma_X^* \upharpoonright \gamma_X^*) \text{ is wellfounded; let } \tilde{\sigma}_X^* : \widehat{M_X^*} \rightarrow \bar{P}_X \text{ be the corresponding ultrapower map.}$
 $\bullet \gamma_0 = \tilde{\sigma}_X^*(\gamma_X^*) < \omega \rho_{\bar{P}_X}^n \text{ and } R_{\bar{P}_X}^n \neq \emptyset.$
 $\bullet \text{ There is a } \Sigma_0^{(n)} \text{ map } \bar{\sigma}_X \text{ from } \bar{P}_X \text{ to } \widehat{M_X} \text{ which extends } \sigma_{0, X}.$

For each $i \in \omega$ and $X \in S$, let $\bar{f}_i^X := \tilde{\sigma}_X^*(f_i^{*X})$ and $u_i^{\bar{P}_X} := u_{\bar{f}_{i+1}^X \in \bar{f}_i^X}^{\bar{P}_X}$. Since $\bar{\sigma}_X$ extends $\sigma_{0, X}$, then for every $i \in \omega$:

$$(6) \quad \bar{\sigma}_X(u_i^{\bar{P}_X}) = \sigma_{0, X}(\tilde{\sigma}_X^*(u_i^{M_X^*})) = \sigma_X^*(u_i^{M_X^*}) = u_i^{M_X}$$

For every $X \in S$, let $\bar{q}_X := \tilde{\sigma}_X^*(q_X^*)$. $\bar{q}_X$ is an element of $R_{\bar{P}_X}^n$ and $\bar{\sigma}_X(\bar{q}_X) = q_X$.

We first take care of the case where $n > 0$.

3.1.1. *When $n > 0$.* For this section, assume $n > 0$. Since $\bar{\sigma}_X$ is $\Sigma_0^{(n)}$ preserving where $n > 0$, then the Π_2 property of being a premouse is preserved downward by $\bar{\sigma}_X$.⁶ So $\bar{P}_X$ is a premouse; let $\bar{M}_X$ denote the bookkeeping premouse of which $\bar{P}_X$ is the expansion. Lemma 14 guarantees that $\bar{M}_X$ is n -iterable.

Claim 35. *There is a pair $X \subset Y$ in S such that:*

- (1) *In the n -coiteration of $\bar{M}_X$ with $\bar{M}_Y$, the $\bar{M}_X$ side is simple*
- (2) $o^{\bar{M}_X}(\mu_0) \leq o^{\bar{M}_Y}(\mu_0)$ ⁷
- (3) $\{\alpha_i^X \mid i \in \omega\} \subseteq \text{rng}(\sigma_Y)$

Proof. Construct a sequence $\langle Z_k \mid k \in \omega \rangle$ as follows: pick Z_0 such that $o^{\bar{M}_{Z_0}}(\mu_0)$ is minimal. Given Z_k , choose Z_{k+1} such that $o^{\bar{M}_{Z_{k+1}}}(\mu_0)$ is minimal among all $Z \in S$ which are “sufficiently large” with respect to each of $Z_0, \dots, Z_k$; i.e. minimal among the set $\{o^{\bar{M}_Z}(\mu_0) \mid Z \in S \text{ and for each } j \leq k, \{\alpha_i^{Z_j} \mid i \in \omega\} \subset \text{range}(\sigma_Z)\}$. This construction ensures that $\langle o^{Z_k}(\mu_0) \mid k \in \omega \rangle$ is monotone increasing. Since the $<^n$ is

⁶Alternatively, the canonical lifting of $\sigma_{0, X}^{-1} \circ \sigma_X^*$ is $\Sigma_2^{(n-1)}$ preserving, so premousehood of M_X^* is preserved upward to $\bar{P}_X$ by this map.

⁷This requirement is irrelevant if CASE (B) holds.

wellfounded, there is some $k^* \in \omega$ such that $\bar{M}_{Z_{k^*+1}} \geq^n \bar{M}_{Z_{k^*}}$. Then the pair $\bar{M}_{Z_{k^*}}, \bar{M}_{Z_{k^*+1}}$ satisfies the requirements of the claim. $\square$

Let $X \subset Y$ be as in Claim 35. Let S denote the last premouse in the n -coiteration of $\bar{M}_X$ with $\bar{M}_Y$, $\bar{\mathcal{I}}_X$ and $\bar{\mathcal{I}}_Y$ the n -iterations on the X, Y sides respectively, and $\bar{\pi}_X$ the n -iteration map on the (simple) $\bar{M}_X$ side. By Lemma 11 the map $\bar{\sigma}_Y : \widehat{\bar{M}_Y} \rightarrow_{\Sigma_0^{(n)}} \widehat{\bar{M}_Y}$ can be used to copy $\bar{\mathcal{I}}_Y$ to a normal n -iteration $\mathcal{I}_Y$ of M_Y , and all the copy maps are still $\Sigma_0^{(n)}$ -preserving. Let S^* be the last iterate of $\mathcal{I}_Y$, and $j^* : \bar{S} \rightarrow_{\Sigma_0^{(n)}} S^*$ be the last copy of $\bar{\sigma}_Y$. The following diagram describes the situation; wavy lines indicate possibly non-simple n -iterations:

$$\begin{array}{ccccc} \bar{M}_Y & \xrightarrow{\bar{\sigma}_Y} & M_Y & & \\ & \searrow \bar{\mathcal{I}}_Y & \searrow \mathcal{I}_Y & & \\ \bar{M}_X & \xrightarrow{\bar{\pi}_X} & \bar{S} & \xrightarrow{j^*} & S^* \end{array}$$

Then for every $i \in \omega$, (6) implies:

$$(7) \quad \langle \alpha_{i+1}^X, \alpha_i^X \rangle \in \sigma_X(u_i^{M_X}) = \sigma_X(\sigma_{0,X}(u_i^{\bar{M}_X}))$$

So applying σ_Y^{-1} yields

$$(8) \quad \langle \sigma_Y^{-1}(\alpha_{i+1}^X), \sigma_Y^{-1}(\alpha_i^X) \rangle \in \sigma_Y^{-1}(\sigma_X(\sigma_{0,X}(u_i^{\bar{M}_X}))) = \sigma_{0,Y}(u_i^{\bar{M}_X}) = j^*(u_i^{\bar{M}_X})$$

The last equality in (8) is due to the fact that the first iteration index of $\bar{\mathcal{I}}_Y$ is $\geq ht(Q_0)$ (because Q_0 is a common initial segment of $\bar{M}_X, \bar{M}_Y$), so $j^* \upharpoonright Q_0 = \bar{\sigma}_Y \upharpoonright Q_0 = \sigma_{0,Y}$.

The remainder of the proof varies slightly, depending on whether CASE (A) or CASE (B) in Lemma 31 holds.

3.1.2. *When CASE (A) holds (i.e. $\mu^{+Q} < \gamma$).* Recall in this case, the hypothesis of Lemma 31 states that there is some $\lambda \in [\mu, \gamma)$ such that λ^{+Q} has uncountable cofinality. So (3) holds at Q_0 ; i.e. $\lambda_X^{+Q_0}$ has uncountable cofinality (in V). Since $\lambda_X \in [\mu_0, \gamma_0)$ and $\bar{M}_X, \bar{M}_Y$ agree below γ_0 , then Corollary (23) implies that $\bar{M}_X$ agrees with $\bar{M}_Y$ below $\min(o^{\bar{M}_X}(\mu_0), o^{\bar{M}_Y}(\mu_0)) = o^{\bar{M}_X}(\mu_0)$; the last equality is due to the fact that X, Y are as in Claim 35. This means that in the n -coiteration of $\bar{M}_X$ with $\bar{M}_Y$:

$$(9) \quad \text{The } \bar{M}_X \text{ side is simple and (strictly) above } \gamma_0$$

(We cannot rule out the possibility that $o^{\bar{M}_X}(\mu_0), o^{\bar{M}_Y}(\mu_0)$ are distinct; i.e. we cannot always guarantee that *both* sides of $\bar{M}_X$ vs. $\bar{M}_Y$ are above γ_0 .)

So $\bar{\pi}_X \upharpoonright Q_0 = id$, and for each $i \in \omega$ we have $u_i^{\bar{M}_X} = \bar{\pi}_X(u_i^{\bar{M}_X})$. This combined with (8) implies that for every $i \in \omega$:

$$(10) \quad \langle \sigma_Y^{-1}(\alpha_{i+1}^X), \sigma_Y^{-1}(\alpha_i^X) \rangle \in j^*(\bar{\pi}_X(u_i^{\bar{M}_X}))$$

Since $j^* \circ \bar{\pi}_X$ is $\Sigma_0^{(n)}$ preserving, the functionally absolute definitions of the good $\widehat{\Sigma_1^{(n-1)}}(\bar{M}_X)$ functions $\bar{f}_i^X$ can be copied via $j^* \circ \bar{\pi}_X$, and so for every $i \in \omega$:

$$(11) \quad j^*(\bar{\pi}_X(u_i^{\bar{M}_X})) = \{ \langle \xi, \eta \rangle \in j^*(\bar{\pi}_X(dom(\bar{f}_{i+1}^X) \cup dom(\bar{f}_i^X))) | j^*(\bar{\pi}_X(\bar{f}_{i+1}^X))(\xi) \in j^*(\bar{\pi}_X(\bar{f}_i^X))(\eta) \}$$

Then (10) and (11) imply $\langle j^*(\bar{\pi}_X(\bar{f}_i^X))(\sigma_Y^{-1}(\alpha_i^X)) | i \in \omega \rangle$ is an infinite decreasing chain of ordinals in $\widehat{S^*}$, a contradiction.

3.1.3. *When CASE (B) holds (i.e. $\mu_0^{+Q_0} = \gamma_0$).* Recall in this case we assumed without loss of generality that the domain of each f_i^X was μ_X . Now (9) may not hold now; but $\bar{M}_X, \bar{M}_Y$ coiterate above μ_0 , so $u_i^{\bar{M}_X} = \bar{\pi}_X(u_i^{\bar{M}_X}) \cap \mu_0$. This, combined with 8, implies:

$$(12) \quad \langle \sigma_Y^{-1}(\alpha_{i+1}^X), \sigma_Y^{-1}(\alpha_i^X) \rangle \in j^*(\bar{\pi}_X(u_i^{\bar{M}_X}) \cap \mu_0)$$

So again we have that $\langle j^*(\bar{\pi}_X(\bar{f}_i^X))(\sigma_Y^{-1}(\alpha_i^X)) | i \in \omega \rangle$ is an infinite decreasing chain of ordinals in $\widehat{S^*}$, a contradiction.

3.1.4. *When $n = 0$.* For this subsection, we assume $n = 0$, so WLOG the $\bar{f}_i^X$ are *elements* of $\bar{P}_X$. If $\bar{P}_X$ is a premouse, then the argument is exactly the same as when $n > 0$; there is a pair X, Y as in Claim 35, where in this case we perform 0-coiterations. Let $\bar{\mathcal{I}}_X, \bar{\mathcal{I}}_Y, \bar{\pi}_X, j^*, S, S^*$ be as in section 3.1.1. In the present case, $j^* \circ \bar{\pi}_X$ is Σ_0 preserving, which is sufficient to obtain a descending chain of ordinals in S^* since the functions $\bar{f}_i^X$ are elements of $\bar{M}_X$.

If $\bar{P}_X$ is a protomouse, let δ_X be maximal such that $E_{top}^{\bar{P}_X}$ is total for $\bar{P}_X \restriction \delta_X = Q_0 \restriction \delta_X$, and let $\tilde{\tau}_X$ be the successor of μ_0 in $Q_0 \restriction \delta_X$. Note that since $|Q_0| < \kappa$, there is a pair $(\delta, \tilde{\tau})$ such that for stationarily many X , $(\tilde{\tau}_X, \delta_X) = (\tilde{\tau}, \delta)$. WLOG assume this holds for every $X \in S$.

Let $\pi : \widehat{\bar{M}_X} \rightarrow_{E_{top}^{\bar{M}_X}} \widehat{\bar{M}'_X}$ and $\widehat{\bar{Q}_X} := \bar{\sigma}_X(\widehat{Q_0 \restriction \delta})$. Let $\widehat{\bar{N}_X} = \pi(\widehat{\bar{Q}_X})$; this is fully iterable. Then $\widehat{\bar{R}_X} := ult(\widehat{Q_0 \restriction \delta_X}, E_{top}^{\bar{P}_X})$ (the premouse associated with $\bar{P}_X$) can be Σ_0 -embedded into the mouse $\widehat{\bar{N}_X}$ via a copy of $(\bar{\sigma}_X \upharpoonright \widehat{Q_0 \restriction \delta_X}, \bar{\sigma}_X \upharpoonright lh(E_{top}^{\bar{P}_X}))$; denote this copy map by j_X . So $\widehat{\bar{R}_X}$ is wellfounded and 0-iterable, and j_X agrees with $\bar{\sigma}_X$ below

$lh(E_{top}^{\bar{P}_X}))$ (the largest cardinal in $\bar{P}_X$). In particular:

$$(13) \quad \text{For every } X, j_X \upharpoonright Q_0 = \sigma_{0,X}$$

Note that since $n = 0$, each $\bar{f}_l^X$ is an element of $\bar{P}_X$ and thus of $\bar{R}_X$. So (13) implies:

$$(14) \quad \text{For every } X \text{ and every } l \in \omega, j_X(u_l^{\bar{R}_X}) = \sigma_{0,X}(u_l^{\bar{P}_X}) = u_l^{M_X}$$

As in Claim (35), there is a pair X, Y in S such that:

- (1) In the 0-coiteration of $\bar{R}_X$ with $\bar{R}_Y$, the $\bar{R}_X$ side is simple
- (2) $o^{\bar{R}_X}(\mu_0) \leq o^{\bar{R}_Y}(\mu_0)$
- (3) $\{\alpha_i^X | i \in \omega\} \subseteq \text{rng}(\sigma_Y)$

The rest of the proof is almost identical to the last section. Copy the R_Y side of the 0-coiteration to a 0-iteration of N_Y (via the map j_Y), let $j^* : \widehat{S} \rightarrow \widehat{S}^*$ be the last copy map, and $\bar{\pi}_X$ the iteration map on the (simple) $\bar{R}_X$ side of the 0-coiteration. The following diagram depicts the situation:

$$(15) \quad \begin{array}{ccccc} \bar{R}_Y & \xrightarrow{j_Y} & N_Y & & \\ & \searrow \bar{\tau}_Y & & \searrow \tau_Y & \\ \bar{R}_X & \xrightarrow{\bar{\pi}_X} & \bar{S} & \xrightarrow{j^*} & S^* \end{array}$$

Then exactly as in the last section, $\langle j^*(\bar{\pi}_X(\bar{f}_i^X))(\sigma_Y^{-1}(\alpha_i^X)) | i \in \omega \rangle$ is a decreasing chain of ordinals in $\widehat{S}^*$, a contradiction.

3.2. Iterability of the ultrapowers. We showed in the last section that for almost every $X \in S$, $\text{ult}^{n_X}(\widehat{M}_X, \sigma_X)$ is wellfounded. Without loss of generality, assume this holds for every $X \in S$. Define:

$$(16) \quad L_X := \text{ult}^{n_X}(\widehat{M}_X, \sigma_X)$$

and for simplicity let σ_X also denote the extension of the original map; so $\sigma_X : \widehat{M}_X \rightarrow_{\Sigma_0^{(n_X)}} L_X$. Due to the fine-structural assumptions on $\widehat{M}_X$, this lifting is n_X -cofinal, $\tilde{\gamma}_X = \sigma_X(\gamma_X)$, $\tilde{\gamma}_X \leq \omega \rho_{L_X}^{n_X}$, and if $n_X > 0$ then $\sigma_X(q_X) \in R_{L_X}^{n_X}$ (recall q_X is our chosen element of $R_{\widehat{M}_X}^{n_X}$).

We need to show that for almost every $X \in S$:

- If L_X is a premouse then it is n_X -iterable
- If L_X is a protomouse whose top predicate is a non-weakly-amenable extender with critical point μ then the usual premouse R_X associated with it is wellfounded and normally m_X -iterable, where $m_X = \text{degree}_{\widehat{R}_X}(\mu)$.

Suppose this fails for stationarily many $X \in S$; without loss of generality we assume it fails for every $X \in S$. Using the κ -completeness of the nonstationary ideal on $P_\kappa(H_\theta)$, without loss of generality we assume that n_X is fixed at some n for every $X \in S$.

We first address the case when L_X is a premouse for stationarily many $X \in S$.

3.2.1. When L_X is a premouse. For this section, assume L_X is a premouse for every $X \in S$. Similarly to Lemma 32, assume WLOG that the current diagram is minimal:

Claim 36. *WLOG we can assume for every $X \in S$: whenever $\zeta_X < \gamma_X$ is a regular cardinal in some premouse R_X such that $\langle R_X, \sigma_X \upharpoonright \zeta_X, \dots | X \in S \rangle$ satisfies the hypothesis of the Frequent Extensions of Embeddings Lemma, then the conclusions of that lemma hold for this system.*

For each $X \in S$:

- (1) Let N_X denote the bookkeeping premouse associated with L_X ; i.e. $L_X = \widehat{N_X}$.
- (2) Fix a countable elementary substructure A_X of H_θ such that $(L_X, \sigma_X(q_X), \dots) \in A_X$, and let $\pi_X^* : H_X^* \rightarrow H_\theta$ be the inverse of the collapsing map for A_X . Let $(L_X^*, N_X^*, \dots) = (\pi_X^*)^{-1}(L_X, N_X, \dots)$. So by elementarity of π_X^* , N_X^* is a (bookkeeping) premouse which is *not* n -iterable. Furthermore, if $n > 0$ then $(\pi_X^*)^{-1}(\sigma_X(q_X)) \in R_{N_X^*}^n$.
- (3) Let $\{z_i^X | i \in \omega\}$ enumerate $\widehat{N_X^*}$.
- (4) Select $\{(f_i^X, \alpha_i^X, r_i^X) | i \in \omega\}$ where $\pi_X^*(z_i^X) = \sigma_X(f_i^X)(\alpha_i^X)$ for every $i \in \omega$, and f_i^X is a good $\Sigma_1^{(n-1)}(\widehat{M_X})$ function in the parameter r_i^X . By the n -cofinality of the map σ_X , if z_i^X is in the n -th reduct of $\widehat{N_X^*}$ then we can take f_i^X to be an element of the n -th reduct of $\widehat{M_X}$.
- (5) Construct countable M_X^* and fully elementary maps $\sigma_X^* : \widehat{M_X^*} \rightarrow \widehat{M_X}$ exactly as section 3.1, except here make sure that $\{r_i^X | i \in \omega\} \subseteq \text{rng}(\sigma_X^*)$.
- (6) Letting $\tilde{\gamma}_X$ be $\sup(\text{range}(\sigma_X^*) \cap \gamma_X)$, note that the supremum of $\sigma_X[\tilde{\gamma}_X]$ is $\geq \sup_{i < \omega} \alpha_i^X$.

Claim 37. *For almost every $X \in S$, σ_X^* maps γ_X^* cofinally to γ_X .*

Proof. Assume this fails for stationarily many $X \in S$, and let $\tilde{\gamma}_X^*$ be the cofinal image of γ_X^* under σ_X^* . Construct $\tilde{P}_X$ and the corresponding

map $\sigma'_X : \tilde{P}_X \rightarrow_{\Sigma_0^{(n)}} \widehat{M}_X$ exactly as in claim 33. Since $\{r_i^X | i \in \omega\} \subseteq \text{rng}(\sigma_X^*)$ then $\tilde{P}_X$ has “copies” of the f_i^X which we’ll denote by $\tilde{f}_i^X$ (for $i \in \omega$); more precisely, $\tilde{f}_i^X$ is the good $\Sigma_1^{(n-1)}(\tilde{P}_X)$ function in the parameter $(\sigma'_X)^{-1}(r_i^X)$ which has the same defin

First suppose that $\tilde{P}_X$ is a premouse for almost every $X \in T$. Then by the minimality assumption of the diagram, the system $\langle \tilde{P}_X, \sigma_X \upharpoonright \tilde{\gamma}_X^* | X \in T \rangle$ satisfies the conclusions of the Frequent Extensions of Embeddings Lemma. Let $\bar{\sigma}_X : \tilde{P}_X \rightarrow_{\Sigma_0^{(n)}} L'_X$ be the lifting guaranteed by the lemma. Then L'_X is a premouse since we are assuming $n > 0$, and L'_X is n -iterable.

Define $j : \widehat{N}_X^* \rightarrow L'_X$ by $j(z_i^X) := \bar{\sigma}_X(\tilde{f}_i^X)(\alpha_i^X)$; this is well-defined by the comments above, i.e., α_i^X really is in the domain of $\bar{\sigma}_X(\tilde{f}_i^X)$. The map j is $\Sigma_0^{(n)}$ preserving, so Lemma 14 guarantees that N_X^* is n -iterable, contrary to our assumptions. That j is $\Sigma_0^{(n)}$ preserving follows from a straightforward computation, using $\Sigma_0^{(n)}$ Los’ Theorem, the fact that σ'_X is $\Sigma_0^{(n)}$ preserving, and the fact that if z_i^X is in the n -th reduct of $\widehat{N}_X^*$ then WLOG we can take f_i^X to be an element of the n -th reduct of $\widehat{M}_X$ (and thus $\tilde{f}_i^X$ is an element of the n -th reduct of $\tilde{P}_X$). A similar computation is given in Claim 38 below.

Now suppose $\tilde{P}_X$ is a protomouse (for stationarily many $X \in T$); this can only happen if $n = 0$. As in the proof of Claim 33, the premouse R_X associated with $\tilde{P}_X$ is wellfounded and satisfies the hypothesis of Claim 36. Let $L'_X = \text{ult}^*(\widehat{R}_X, \sigma_X \upharpoonright \tilde{\gamma}_X^*)$.

L'_X is a premouse: otherwise $\mu_X^{+\widehat{R}_X} < \tilde{\gamma}_X^*$ and $\sigma_X \upharpoonright \tilde{\gamma}_X^*$ maps this ordinal non-continuously. But then $\mu_X^{+\widehat{R}_X} = \mu_X^{+\widehat{M}_X}$ (since $cr(\sigma'_X) = \tilde{\gamma}_X^*$) and so $\tilde{\sigma}_X$ (which extends σ_X) would yield that L_X is a protomouse, contrary to the assumption in this section.

Let $\bar{\sigma}_X : \widehat{R}_X \rightarrow \widehat{L}'_X$ be the canonical lifting. Finally, construct a map $j : \widehat{N}_X^* \rightarrow \widehat{L}'_X$ as in the earlier part of this proof (note that the earlier definition of j still makes sense here, because $\tilde{f}_i^X$ is an *element* of $\tilde{P}_X$ and thus of R_X). It is straightforward to check that this map is Σ_0 preserving, which contradicts the fact that N_X^* is not 0-iterable. $\square$

Using the same arguments as in section 3.1, there is a Q_0 such that for stationarily many X , $\sigma_{0,X}^{-1} \circ \sigma_X^*$ is a total Σ_0 cofinal map from $Q_X^* \rightarrow Q_0$. For such X , as in Lemma (34) construct $\bar{\sigma}_X : \bar{P}_X \rightarrow_{\Sigma_0^{(n)}} \widehat{M}_X$, where $\bar{P}_X = \text{ult}^n(M_X^*, \sigma_{0,X}^{-1} \circ \sigma_X^*)$ and $\bar{\sigma}_X$ extends $\sigma_{0,X}$.

As in section 3.1, we break the proof into cases according to the value of n .

3.2.2. *When $n > 0$.* Since we're assuming $n > 0$, $\bar{\sigma}_X$ is Σ_2 preserving and $\bar{P}_X$ is a premouse. Let $\bar{M}_X$ be its associated bookkeeping premouse; i.e. $\bar{P}_X = \widehat{\bar{M}_X}$. Lemma 14 guarantees that $\bar{M}_X$ is n -iterable.

Let $\bar{f}_i^X$ denote the “copy” of f_i^X over $\widehat{\bar{M}_X}$; more precisely, $\bar{f}_i^X$ is the good $\Sigma_1^{(n-1)}(\widehat{\bar{M}_X})$ function in the parameter $(\bar{\sigma}_X)^{-1}(r_X)$ which has the same definition as f_i^X has over $\widehat{M_X}$ (in the parameter r_X). As in Claim (35) there is a pair pair $X \subset Y$ in S such that:

- In the n -coiteration of $\bar{M}_X$ with $\bar{M}_Y$, the $\bar{M}_X$ side is simple
- $o^{\bar{M}_X}(\mu_0) \leq o^{\bar{M}_Y}(\mu_0)$ ⁸
- $\{\alpha_i^X | i \in \omega\} \subseteq \text{rng}(\sigma_Y)$

Let $\bar{\mathcal{I}}_X, \bar{\mathcal{I}}_Y$ denote the the $\bar{M}_X, \bar{M}_Y$ sides of their n -coiteration (respectively), let $\bar{\pi}_X : \widehat{\bar{M}_X} \rightarrow \widehat{S}$ denote the iteration map on the (simple) $\bar{M}_X$ side, $\mathcal{I}_Y$ be the copy of $\bar{\mathcal{I}}_Y$ to M_Y via $\bar{\sigma}_Y$ which exists by Lemma 11, and $j^* : \widehat{S} \rightarrow \widehat{S^*}$ be the last copy map. Define $k : \widehat{N_X^*} \rightarrow \widehat{S^*}$ by $z_i^X \mapsto j^* \circ \bar{\pi}_X(\bar{f}_i^X)(\sigma_Y^{-1}(\alpha_i^X))$. In section 3.1 we found a decreasing chain of ordinals in S^* ; but here we will use S^* and k to show that N_X^* is n -iterable to yield a contradiction and complete the proof for section 3.2.2. The situation is depicted in the following diagram, where wavy lines indicate possibly non-simple iterations:

$$\begin{array}{ccccc}
 \bar{M}_Y & \xrightarrow{\bar{\sigma}_Y} & M_Y & & \\
 & \searrow \bar{\mathcal{I}}_Y & \swarrow \mathcal{I}_Y & & \\
 \bar{M}_X & \xrightarrow{\bar{\pi}_X} & S & \xrightarrow{j^*} & S^* \\
 & & & \uparrow k & \\
 & & & N_X^* &
 \end{array}$$

Claim 38. k is $\Sigma_0^{(n)}$ preserving.

Proof. By Lemma 11 the copy map j^* is $\Sigma_0^{(n)}$ preserving and agrees with $\bar{\sigma}_Y$ below the first coiteration index ν_0 . Since Q_0 is a common initial segment of $\bar{M}_X, \bar{M}_Y$ then $\nu_0 \geq \gamma_0$, so

$$(17) \quad \bar{\sigma}_Y \upharpoonright Q_0 = j^* \upharpoonright Q_0 = \sigma_{0,Y}$$

First, we show that $k[H_{N_X^*}^n] \subseteq H_{S^*}^n$. Let $z_i^X \in H_{N_X^*}^n$; then $\pi_X^*(z_i^X) \in H_{N_X}^n$ and since σ_X is n -cofinal, then we can assume $f_i^X \in H_{\bar{M}_X}^n$. Thus

⁸This requirement is irrelevant if CASE (B) holds.

$\bar{f}_i^X \in H_{\widehat{M_X}}^n$ and so $j^* \circ \bar{\pi}_X(\bar{f}_i^X) \in H_{\widehat{S^*}}^n$. Now $\alpha_i^X < \sigma_X(\text{dom}(\bar{f}_i^X)) = \sigma_Y \circ \sigma_{XY}(\text{dom}(\bar{f}_i^X))$, so:

$$(18) \quad \sigma_Y^{-1}(\alpha_i^X) < \sigma_{XY}(\text{dom}(\bar{f}_i^X)) = \sigma_{XY} \circ \sigma_{0X}(\text{dom}(\bar{f}_i^X)) = \sigma_{0Y}(\text{dom}(\bar{f}_i^X)) = j^*(\text{dom}(\bar{f}_i^X))$$

(the last equality is due to (17))

If CASE (A) holds, then as section 3.1 the $\bar{M}_X$ side of the coiteration is above γ_0 (see (9)) and $\gamma_0 > \text{dom}(\bar{f}_i^X)$; so in this case $j^*(\text{dom}(\bar{f}_i^X)) = j^* \circ \bar{\pi}_X(\text{dom}(\bar{f}_i^X))$. Then by (18), $\sigma_Y^{-1}(\alpha_i^X)$ is in the domain of $j^* \circ \bar{\pi}_X(\bar{f}_i^X)$; and since this latter function is an element of the n -th reduct of $\widehat{S^*}$, then $k(z_i^X) = j^* \circ \bar{\pi}_X(\bar{f}_i^X)(\sigma_Y^{-1}(\alpha_i^X))$ is an element of the n -th reduct of $\widehat{S^*}$.

If CASE (B) holds, then recall $\text{dom}(\bar{f}_i^X) = \mu_0$ for every $i \in \omega$. Since the coiteration of $\bar{M}_X, \bar{M}_Y$ is above μ_0 we have $j^*(\text{dom}(\bar{f}_i^X)) = j^*(\mu_0 \cap \bar{\pi}_X(\text{dom}(\bar{f}_i^X))) = j^*(\mu_0) \cap \text{dom}(j^* \circ \bar{\pi}_X(\bar{f}_i^X)) = \mu_0 \cap \text{dom}(j^* \circ \bar{\pi}_X(\bar{f}_i^X))$ (note $\mu_0 < cr(j^*)$ by 17). So again by 18 we have that $\sigma_Y^{-1}(\alpha_i^X)$ is in the domain of $j^* \circ \bar{\pi}_X(\bar{f}_i^X)$ and $k(z_i^X)$ is an element of the n -th reduct of $\widehat{S^*}$. This completes the proof that $k[H_{\widehat{N_X}}^n] \subseteq H_{\widehat{S^*}}^n$.

Now suppose Φ is a $\Sigma_0^{(n)}$ formula, and let $z_{i_1}^X, \dots, z_{i_l}^X$ be elements of $\widehat{N_X^*}$. Let f'_i denote the function $j^* \circ \bar{\pi}_X(\bar{f}_i^X)$. Then:

$$(19) \quad \begin{aligned} \widehat{N_X^*} \models \Phi(z_{i_1}^X, \dots, z_{i_l}^X) & \iff \widehat{N_X} \models \Phi(\pi_X^*(z_{i_1}^X), \dots, \pi_X^*(z_{i_l}^X)) \\ & \iff \widehat{N_X} \models \Phi(\sigma_X(f_{i_1}^X)(\alpha_{i_1}^X), \dots, \sigma_X(f_{i_l}^X)(\alpha_{i_l}^X)) \\ & \iff \prec \alpha_{i_1}^X, \dots, \alpha_{i_l}^X \succ \in \sigma_X(u_{\Phi, \bar{f}^X}^{M_X}) \\ & \iff \prec \sigma_Y^{-1}(\alpha_{i_1}^X), \dots, \sigma_Y^{-1}(\alpha_{i_l}^X) \succ \in \sigma_Y^{-1} \circ \sigma_X(u_{\Phi, \bar{f}^X}^{M_X}) \\ & = \sigma_{XY}(u_{\Phi, \bar{f}^X}^{M_X}) = \sigma_{XY} \circ \bar{\sigma}_X(u_{\Phi, \bar{f}^X}^{\bar{M}_X}) \\ & = \sigma_{XY} \circ \sigma_{0,X}(u_{\Phi, \bar{f}^X}^{\bar{M}_X}) = \sigma_{0,Y}(u_{\Phi, \bar{f}^X}^{\bar{M}_X}) \\ & = \bar{\sigma}_Y(u_{\Phi, \bar{f}^X}^{\bar{M}_X}) = j^*(u_{\Phi, \bar{f}^X}^{\bar{M}_X}) \end{aligned}$$

If CASE (A) holds, then $cr(\bar{\pi}_X) \geq \gamma_0$ and $\gamma_0 > \text{dom}(\bar{f}_i^X)$ for every $i \in \omega$, so $j^*(u_{\Phi, \bar{f}^X}^{M_X}) = j^* \circ \bar{\pi}_X(u_{\Phi, \bar{f}^X}^{\bar{M}_X}) = u_{\Phi, \bar{f}'}^{\widehat{S^*}}$. Combined with (19), this implies that $\widehat{N_X^*} \models \Phi(z_{i_1}^X, \dots, z_{i_l}^X) \iff \widehat{S^*} \models \Phi(k(z_{i_1}^X), \dots, k(z_{i_l}^X))$.

If CASE (B) holds, then $cr(\bar{\pi}_X) \geq \mu_0$ and $\mu_0 = \text{dom}(\bar{f}_i^X)$ for every $i \in \omega$, so $j^*(u_{\Phi, \bar{f}^X}^{\bar{M}_X}) = j^*(\mu_0 \cap u_{\Phi, \bar{f}^X}^{\bar{M}_X}) = j^*(\mu_0) \cap u_{\Phi, \bar{f}'}^{\widehat{S^*}} = \mu_0 \cap u_{\Phi, \bar{f}'}^{\widehat{S^*}}$.

Combined with (19), this implies that $\widehat{N_X^*} \models \Phi(z_{i_1}^X, \dots, z_{i_l}^X) \iff \widehat{S^*} \models \Phi(k(z_{i_1}^X), \dots, k(z_{i_l}^X))$. $\square$

Lemma 14 and Claim 38 imply N_X^* is n -iterable, which is a contradiction. This completes the proof in section 3.2.2.

3.2.3. *When $n = 0$.* We are still assuming L_X is a premouse; but now $\bar{P}_X$ might be a protomouse. As in section 3.1.4, the premouse $\bar{R}_X$ associated with $\bar{P}_X$ is wellfounded, and again we obtain Diagram 15. Since $n = 0$ then the functions $\bar{f}_i^X$ are elements of $\bar{P}_X$ and thus of $\bar{R}_X$. Similarly to section 3.2.2 N_X^* can be Σ_0 -embedded into the premouse S^* at the end of Diagram 15, which contradicts the fact that N_X^* is not 0-iterable.

3.2.4. *When L_X is a protomouse.* For this section we assume L_X is a protomouse for some $X \in S$ (recall $L_X = \text{ult}^{n_X}(\widehat{M}_X, \sigma_X)$ was defined at (16)). This implies that $n = 0$, since otherwise σ_X would be Σ_2 -preserving and the premousehood of $\widehat{M}_X$ would carry upward to L_X .

Since L_X is not a premouse—i.e. its top extender F_X is not total—let δ_X be maximal such that F_X is total on $L_X \restriction \delta_X = Q \restriction \delta_X$. Let $\tau_X := \mu_X^{+Q_X}$. Then σ_X does not map τ_X cofinally to $\sigma_X(\tau_X)$ (otherwise F_X would be total on L_X). In particular, the continuity of $\sigma_X \restriction Q_X$ on $\text{cof}(\omega)$ (see (2)) implies that τ_X and $\tilde{\tau}_X := \sup(\sigma_X[\tau_X]) = \mu^{+Q \restriction \delta_X}$ have uncountable cofinality.

μ does not index an extender in Q since μ is measurable in Q , and $Q \restriction \delta_X$ is sound above μ . Then by Lemma 16, the premouse R_X associated with L_X (see Definition 25) is wellfounded and fully m -iterable, where $m := \text{degree}_{\widehat{Q \restriction \delta_X}}(\mu)$. This completes the proof of Lemma 31.

4. CONSEQUENCES OF THE FREQUENT EXTENSIONS OF EMBEDDINGS LEMMA

Lemma 39, whose argument is due to Mitchell, is key to the proof of the well-known Weak Covering Lemma (Lemma 42) below. We note that once the Weak Covering Lemma has been proved, then Lemma 39 can be strengthened to obtain Lemma 44 in cases where $X \cap \text{Ord}$ is closed under limits of countable cofinality.

Lemma 39. *Assume 0-pistol does not exist, and let K be the core model. Let S be a stationary subset of weakly internally approachable sets in $P_\kappa(H_\chi)$ where $\kappa \geq \omega_2$ and χ is a large regular cardinal. For $X \in S$ let $\pi_X : H_X \rightarrow X \prec (H_\chi, \in, K \cap H_\chi, \dots)$ be the inverse of the transitive collapsing map, and $K_X := \pi_X^{-1}[K \cap H_\chi]$. Let $\alpha_X = X \cap \kappa = \text{cr}(\pi_X)$ and $\sigma_X := \pi_X \upharpoonright K_X$.*

For the coiteration of K_X with K , let

$$\langle M_i^X, \pi_{i,j}^{M^X}, N_i^X, \pi_{i,j}^{N^X}, \nu_i^X, \kappa_i^X, \tau_i^X, \delta_i^X, (N_i^X)^* \mid i \leq j \leq \theta_X \rangle$$

denote the coiteration data as in Definition 9, where $K = N_0^X$ and $K_X = M_0^X$. Then for almost every $X \in S$:

- (1) K_X does not move in the coiteration; i.e. there are no extenders applied on the K_X side of the coiteration.
- (2) $\alpha_X^{+K_X}$ is not a cardinal in K .⁹
- (3) If ν_0^X is defined (i.e. K_X is not an initial segment of K) then $\nu_0^X < \alpha_X^{+K}$.
- (4) The K side truncates by stage 1. There are two possibilities:
CASE (A): $\tau_0^X \geq \alpha_X$ and K truncates at stage 0, or
CASE (B): $\tau_0^X < \alpha_X$ and $(\nu_0^X)^{+K_X}$ is not a cardinal in $K_1 := \text{ult}(K, E_{\nu_0^X})$.¹⁰
- (5) Let $\iota_0^X = 0$ if CASE (A) holds and $\iota_1^X = 1$ if CASE (B) holds; so ι_0^X is the first truncation stage. Then for every $j \geq \iota_0^X$: the ultimate projectum of $(N_j^X)^*$ is $\leq \kappa_j^X$ and $(N_j^X)^*$ is sound above κ_j^X .

Here is a sketch of the proof; more details can be found in [10].

Proof. Suppose that $\alpha_X^{+K_X}$ were a cardinal in K for stationarily many $X \in S$. For such X it makes sense to form $\text{ult}(K, \sigma_X \upharpoonright \alpha_X^{+K_X})$; and by Lemma 31, this ultrapower is wellfounded for almost every such X .

⁹So by the convention of Definition 9, we define $\kappa_0^X, \tau_0^X, \delta_0^X$, and $(N_0^X)^*$ even if K_X is an initial segment of K .

¹⁰Then by the convention in Definition 9, the objects $\kappa_1^X, \tau_1^X, \delta_1^X$, and $(N_1^X)^*$ are defined even if K_X is an initial segment of K_1 (if that happens here, then $\kappa_1^X = \nu_0^X$).

Let $\tilde{\sigma}_X : K \rightarrow K'_X$ denote the corresponding lifting. By Lemma 28, $\tilde{\sigma}_X$ is a simple normal iteration map of K with critical point α_X ; let ν be the first iteration index. So ν indexes an extender in K but not in K'_X . By Lemma 7 and the normality of the iteration, $\nu < \tilde{\sigma}_X(\alpha_X)$. This is a contradiction, since $\tilde{\sigma}_X(\alpha_X) = \sigma_X(\alpha_X) = \kappa$ and K'_X end-extends $K|sup(\sigma_X[\alpha_X^{+K_X}])$ (because K'_X was defined as the ultrapower associated with the canonical lifting of $\sigma_X \upharpoonright \alpha_X^{+K_X}$ to K).

Item (3) is a simple consequence of item (2) and coherency. If ν_0^X were $\geq \alpha_X^{+K}$, then $\alpha_X^{+K} = \alpha_X^{+N_1^X} = \alpha_X^{+M_1^X} = \alpha_X^{+K_X}$, which contradicts item (2).

Now we prove (4). If $\tau_0^X = (\kappa_0^X)^{+K_X} = \alpha_X$ then clearly τ_0^X is not a cardinal in K , since $\alpha_X = cr(\sigma_X)$. If $\tau_0^X > \alpha_X$ then (2) implies that τ_0^X is not a cardinal in K . For the remaining case $\alpha_X > \tau_0^X$ we use the following claim:

Claim 40. *If $\alpha_X > \tau_0^X$ (so there is no truncation at stage 0), then for almost every $X \in S$:*

- ν_0^X does not index an extender in K_X
- $(\nu_0^X)^{+K_X}$ is not a cardinal in $N_1^X = ult(K, E_{\nu_0^X}^{K_X})$ (this implies that the K side truncates at stage 1, since $\kappa_1^X > \nu_0^X$)

Proof. Suppose that ν_0^X did index an extender in K_X ; let j_X be the ultrapower map for $ult(K_X, E_{\nu_0^X}^{K_X})$. Then since $\tau_0^X < \alpha_X$ is a cardinal in K , Lemma 31 guarantees that $ult(K, j_X \upharpoonright \tau_0^X)$ is wellfounded for almost every $X \in S$. It follows easily that $E_{\nu_0^X}^{K_X}$ is K -correct and is on K 's extender sequence by Lemma 28, contradicting the fact that ν_0^X is a coiteration index.

To see that $(\nu_0^X)^{+K_X}$ is not a cardinal in N_1^X for almost every $X \in S$: suppose it were. By the first part of this claim and the coherency of the extender $E_{\nu_0^X}^{K_X}$, then $E_{\nu_0^X}^{K_X} = \emptyset$ and so $K_X|(\nu_0^X)^{+K_X}$ is an initial segment of N_1^X . Using Lemma 31 lift $\pi_X \upharpoonright (\nu_0^X)^{+K_X}$ to N_1^X and let $\tilde{\pi}_X : N_1^X \rightarrow L^X$ be this lifting. Then $\tilde{\pi}_X \circ \pi_{0,1}^X$ is a map from $K \rightarrow N_1^X \rightarrow L_X$, where $\pi_{0,1}^X$ is the ultrapower map for the first step of the K side (i.e. $\pi_{0,1}^X : K \rightarrow_{E_{\nu_0^X}^{K_X}} N_1^X$). By Lemma 28, the map $\tilde{\pi}_X \circ \pi_{0,1}^X$ must be a simple normal iteration map, and the first part of it—to N_1^X —used the index ν_0^X . So normality of this iteration requires that all critical points in the part from N_1^X to L_X must be strictly above ν_0^X . But the critical point of $\tilde{\pi}_X$ is α_X , and $\alpha_X \leq \nu_0^X$. This is a contradiction. $\square$

Item 5 then follows by a straightforward induction on $\iota \geq \iota_0^X$; this is a general fact about a normal linear iteration after a truncation has occurred.

Finally we prove (1). Suppose this fails, and let ι_X be the first stage where the K_X side applies an extender. By Claim 40, we can assume that $\iota_X \geq \iota_0^X$. By applying the Fodor Lemma for $P_\kappa(H_\theta)$, we can WLOG assume that $\pi_X(\nu_{\iota_X}^X)$ is the same for every $X \in S$; let ν' denote this common value. Let $j : K \rightarrow_{E_{\nu'}^K} K'$ and let $\kappa' = cr(E_{\nu'}^K)$.

For each X let:

- $Q_X := K_X | \tau_{\iota_X}^X$
- $\sigma_X := j \circ \pi_X \upharpoonright Q_X$.
- $\bar{F}_X := E_{\nu_{\iota_X}^X}^{K_X}$, $\bar{\kappa}_X := cr(\bar{F}_X)$, and $\bar{\tau}_X = \tau_{\iota_X}^X = \bar{\kappa}_X^{+K_X}$
- $N_X^* := (N_{\iota_X}^X)^*$
- $n_X := \deg_{\widehat{N_X^*}}(\kappa_{\iota_X}^X)$; WLOG we will assume this has a fixed value n for every $X \in S$.

Since there is an extender on $\kappa_{\iota_X}^X$ in K_X , then Lemma 7 implies that $o^{K_X}(\mu) = o^{N_X^*}(\mu) < \kappa_{\iota_X}^X$ for every $\mu < \kappa_{\iota_X}^X$. Then the canonical lifting of σ_X guaranteed by Lemma 31 yields a premouse which end-extends $K | j(\kappa')$ and is n -iterable; the fact that this lifting yields a premouse (and not a protomouse) follows from Lemma 7. Let $\tilde{\sigma}_X : \widehat{N_X^*} \rightarrow \widehat{N_X'}$ be this canonical lifting. Also, since $\widehat{N_X^*}$ is sound above $\kappa_{\iota_X}^X$ then $\widehat{N_X'}$ is sound above $j(\kappa) = \tilde{\sigma}_X(\bar{\kappa}_X)$ and $\omega \rho_{\widehat{N_X'}}^{n+1} \leq j(\kappa')$.¹¹

N_X' is in fact fully iterable: N_X' and K' agree below $j(\kappa')$ and $j(\kappa')$ does not index an extender in either, so their attempted *-coiteration is above $j(\kappa') \geq \omega \rho_{\widehat{N_X'}}^{n+1}$. Thus the n -iterable premouse N_X' is *-coiterable with K' and so N_X' is fully iterable (since K' is universal and thus the N_X' side of the coiteration is simple).

Now $\bar{F}_X$ is on the sequence of the mouse K_X and so is weakly amenable with respect to $K_X | \bar{\tau}_X = N_X^* | \bar{\tau}_X$. Define a map $k : ult^*(\widehat{N_X^*}, \bar{F}_X) \rightarrow \widehat{N_X'}$ by $[\beta, f]_{\bar{F}_X} \mapsto \tilde{\sigma}_X(f)(\pi_X(\beta))$,¹² where $\beta < lh(\bar{F}_X) = \nu_{\iota_X}^X$ and f is a good $\Sigma_1^{(n-1)}(\widehat{N_X^*})$ function. It is straightforward to check that that k is well-defined, $\Sigma_0^{(n)}$ preserving, and n -cofinal. Let $\widehat{\tilde{N}_X} := ult^*(\widehat{N_X^*}, \bar{F}_X)$

¹¹Alternatively, one could first lift N_X^* to an initial segment of K , take the ultrapower of that lifting by $E_{\nu'}$, and then embed $\tilde{N}_X$ into that ultrapower; this amounts to the same thing.

¹²Recall π_X is the map from $K_X \rightarrow K$ and $\tilde{\sigma}_X$ extends $j \circ \pi_X \upharpoonright \bar{\tau}_X$.

and $\pi_{\bar{F}_X}$ be the corresponding ultrapower map. This map is $\Sigma_0^{(n)}$ preserving and n -cofinal and $\pi_{\bar{F}_X}(p) \in R_{\widehat{N}_X}^n$ where p is the standard parameter of $\widehat{N}_X^*$ above the n -th projectum. Lemma 14 guarantees that $\tilde{N}_X$ is n -iterable and that in the n -coiteration of N_X^* with $\tilde{N}_X$, the N_X^* side is simple. Also, it is easy to check that the critical points of this coiteration are $\geq \bar{\kappa}_X$ which in turn is $\geq$ the $(n+1)$ st projecta of both N_X^* and $\tilde{N}_X$; so this n -coiteration is really just the $*$ -coiteration.

Claim 41. N_X^* and $\tilde{N}_X$ coiterate simply to a common mouse.

Proof. We have already seen that the N_X^* side is simple. Since $\pi_{\bar{F}_X}$ can be viewed as a lifting of the ultrapower of $K_X|\bar{\tau}_X$ by $\bar{F}_X$, then:

$$(20) \quad \tilde{N}_X \text{ agrees with } M_{\iota_X+1}^X \text{ below } \pi_{\bar{F}_X}(\kappa_{\iota_X}^X)$$

$$(21) \quad \nu_{\iota_X}^X \text{ does not index an extender in } \tilde{N}_X \text{ (by the coherency of } \bar{F}_X \text{)}.$$

CASE 1: ν_{ι_X} indexes an extender in N_X^* .

In this case the N_X^* side applies $E_{\nu_{\iota_X}^X}^{N_X^*}$ at the first step of the coiteration; let N^{**} denote the resulting mouse. Then:

- (1) N^{**} and $\tilde{N}_X$ coiterate above $\nu_{\iota_X}^X$
- (2) N^{**} and $\tilde{N}_X$ have the same successor of $\nu_{\iota_X}^X$
- (3) The expansions of N^{**} and $\tilde{N}_X$ are both sound above $\nu_{\iota_X}^X$

Item (2) is due to the fact that the K_X side of the K vs. K_X coiteration is simple, while at the same time the N_X^* side of the N_X^* vs. $\tilde{N}_X$ coiteration is simple.

Standard arguments then show that N^{**} and $\tilde{N}_X$ coiterate simply to a common mouse, and that in fact $N^{**} = \tilde{N}_X$.

CASE 2: ν_{ι_X} does not index an extender in N_X^* .

In this case, both sides of the coiteration N_X^* vs. $\tilde{N}_X$ are above $\nu_{\iota_X}^X$. Similarly to Case 1, N_X^* and $\tilde{N}_X$ have the same successor of $\nu_{\iota_X}^X$, and the argument is similar to Case 1. $\square$

Let R be the common simple iterate from Claim 41, $k : \widehat{N}_X^* \rightarrow \hat{R}$ be the iteration map on the N_X^* side, and $\tilde{k} : \widehat{\tilde{N}_X} \rightarrow \hat{R}$ be the iteration map on the $\tilde{N}_X$ side. Note that $cr(\tilde{k}) > \bar{\kappa}$ by (20) and (21). The soundness of $\widehat{N}_X^*$ above $\kappa_{\iota_X}^X$, along with the preservation degree of $\pi_{\bar{F}_X}$, $\tilde{k}$, and k , guarantees that $\tilde{k} \circ \pi_{\bar{F}_X} = k$; so in fact $\nu_{\iota_X}^X$ must index an extender in N_X^* (i.e. Case 2 from Claim 41 is impossible). The commutivity of the diagram, and the fact that $cr(\tilde{k}) > \kappa_{\iota_X}^X$, imply that for every $\eta < \nu_{\iota_X}^X$

and $z \in K_X|_{\tau_X}$:

$$(22) \quad \eta \in \bar{F}_X(z) \iff \eta \in E_{\nu_{\iota_X}^X}^{N_X^*}(z)$$

In other words, $E_{\nu_{\iota_X}^X}^{N_X^*}$ and $\bar{F}_X$ are the same extender. This is a contradiction, since $\nu_{\iota_X}^X$ was a coiteration index. $\square$

The following two lemmas are proved by a joint induction; the induction assumption guarantees that requirement (1) of Lemma 31 is satisfied when Q is an initial segment of K .

Lemma 42. (*Weak Covering Lemma*) Assume 0-pistol does not exist, and let K be the core model. If $\kappa \geq \omega_2$ then $cf(\kappa^{+K}) \geq |\kappa|$.

Proof. The argument relies heavily on Lemma 39. A detailed proof of Weak Covering for the core model for measures of order zero appears at Theorem 7.5.1 in [10]; on page 278 he points out the minor revisions needed to prove Weak Covering for the core model below 0-pistol. $\square$

Lemma 43. Consider the special case of Lemma 31 where Q is an initial segment of the core model K , γ is a cardinal in K , $n_X = \text{degree}_{\widehat{M_X}}(\gamma_X)$, and $\widehat{M_X}$ is sound above γ_X . Then $L_X = \text{ult}^*(\widehat{M_X}, \sigma_X) = \text{ult}^{n_X}(\widehat{M_X}, \sigma_X)$ is wellfounded.

If L_X is a premouse, then letting N_X be its associated bookkeeping premouse (i.e. $L_X = \widehat{N_X}$):

- If $o^K(\mu) < \gamma$ for every $\mu < \gamma$, then N_X is an initial segment of K
- If there is some $\mu < \gamma$ such that $\gamma \leq o^K(\mu)$, then $o^{N_X}(\mu) \leq o^K(\mu)$. If $o^{N_X}(\mu) = o^K(\mu)$, then N_X is an initial segment of K . If $o^{N_X}(\mu) < o^K(\mu)$ then N_X is an initial segment of $\text{ult}(K, E_{o^{N_X}(\mu)})$.

If L_X is not a premouse, let R_X be the premouse associated with L_X (see Definition 25). Let F_X be the non-weakly-amenable top extender of L_X , $\mu = cr(F_X)$, and $\lambda_X = lh(F_X)$ (= the largest cardinal in L_X). Then $o^K(\mu) \geq o^{R_X}(\mu)$, and:

- If $o^K(\mu) = o^{R_X}(\mu)$ then R_X is an initial segment of K
- If $o^K(\mu) > o^{R_X}(\mu)$ then R_X is an initial segment of $\text{ult}(K, E_{o^{R_X}(\mu)}^K)$.

Proof. Suppose first that L_X is a premouse and N_X is its cutback. The soundness of $\widehat{M_X}$ above γ_X implies that L_X is sound above γ (and that $\omega\rho_{L_X}^{n_X+1} \leq \gamma$). If γ does not index an extender in K , or it indexes an extender on some $\mu < \gamma$ and $o^{N_X}(\mu) = o^K(\mu)$, then N_X and K coiterate above γ . It follows that N_X is an initial segment of K .

If $o^{N_X}(\mu) < o^K(\mu)$, then at the first stage the K side applies the extender $E_{o^{N_X}(\mu)}^K$; let $K_1 = \text{ult}(K, E_{o^{N_X}(\mu)}^K)$. Then K_1 and N_X coiterate above γ , and N_X is an initial segment of K_1 .

We can rule out the case that $o^{N_X}(\mu) > o^K(\mu)$: the inductive assumption that (1) of Lemma 31 holds in Q , along with Corollary 19, implies that any extender on μ in N_X would be K -correct. Since K absorbs K -correct extenders (see Lemma 28), then such an extender would be on K 's sequence.

Now suppose that L_X is not a premouse; i.e. its top predicate F_X is not weakly amenable. As in section 3.2.4, the premouse $\widehat{R}_X$ associated with L_X (see Definition 25) exists and is fully m_X -iterable. Let $\pi : \widehat{L_X} \restriction \delta \rightarrow_{F_X}^* \widehat{R}_X$. By Lemma 26, $\widehat{R}_X$ is sound above the sup of generators of F_X , which is in turn $\leq o^{R_X}(\mu)$. As in the case where L_X was a premouse, we can rule out the case that $o^{R_X}(\mu) > o^K(\mu)$. So in the coiteration of R_X with K , the R_X side is above $o^{R_X}(\mu)$. The rest of the argument is similar to the case when L_X was a premouse, so R_X is either an initial segment of K or of K_1 . Finally, we note that the protomouse L_X is computable from $\widehat{R}_X$ (see Lemma 26), so $L_X \in K$ (in fact an element of $K \restriction \gamma^{+K}$, since L_X is sound above γ). $\square$

The following is a different version of Lemma 39. Its proof is similar to the proof of Lemma 39, except that Lemmas 16, 17, and 42 are used instead of the Frequent Extensions of Embeddings Lemma:

Lemma 44. *Assume $X \prec (H_\theta, \in, \dots)$ and $X \cap \text{Ord}$ is ω -closed. Let $\pi_X : H_X \rightarrow X$ be the inverse of the transitive collapse map, $\alpha = \text{cr}(\pi_X)$, and K_X the core model constructed in H_X . Assume $\sigma_X(\alpha_X) \geq \omega_2$. Then the conclusions of Lemma 39 hold for the K vs. K_X coiteration.*

Note that we do not place any cardinality restrictions on X . For example, X might be an ω_2 -sized structure witnessing $(\omega_3, \omega_2) \twoheadrightarrow (\omega_2, \omega_1)$.

5. COVERING THEOREMS

In this section we prove Theorems 45 and Theorem 46.

Theorem 45. *Assume $\gamma > \omega_2$, $cf(\gamma) < |\gamma|$, and γ is regular in K . Then γ is measurable in K .*

If $cf(\gamma)$ is uncountable, the consequence is stronger:

Theorem 46. *Assume $\gamma > \omega_2$, $\omega < cf(\gamma) < |\gamma|$, and γ is regular in K . Then:*

- (1) *For every $\beta < cf(\gamma)$, $\{\lambda < \gamma \mid o_*^K(\lambda) \geq \beta\}$ contains a closed unbounded set in γ .¹³*
- (2) *$o_*^K(\gamma) \geq cf(\gamma)$.*

Part 2 of Theorem 46 is really a consequence of part 1 and the following theorem:

Theorem 47. *Assume $\gamma > \omega_2$, $\omega < cf(\gamma) < |\gamma|$, and γ is regular in K . Let $a \in P^K(\gamma)$ and let S_a be the collection of $\lambda < \gamma$ such that K has a normal measure on λ which concentrates on $a \cap \lambda$. Assume S_a is stationary in V . Then K has a normal measure on γ which concentrates on a .*

Theorem 47 will also be used in section 6.

We note the following, although it is not used in the proof of any of the theorems above:

Note 48. *If γ is as in the hypothesis of Theorem 45 then*

- (1) *γ is inaccessible in K*
- (2) *either:*
 - *γ is a singular cardinal in V , or*
 - *$\gamma \in (\kappa, \kappa^+)$ for some κ which is regular in V .*

Proof. To see that γ is inaccessible in K , suppose otherwise; say $\gamma = \lambda^{+K}$. Then if γ is not a cardinal in V , by Lemma 42 (the weak covering lemma) $cf(\gamma) \geq |\lambda| = |\gamma|$, contrary to our assumption. If $\gamma = \lambda^{+K}$ is also a cardinal in V then it must be a successor cardinal in V , again contrary to our assumption that γ is singular in V .

To see why (2) must hold: if γ is *not* a cardinal in V , then $|\gamma|$ must be regular in V , since K correctly computes the successors of singular cardinals in V by Lemma 42 (the weak covering lemma). $\square$

¹³Recall from Definition 6 that for mice below 0-pistol, $o_*^M(\kappa)$ denotes the “Mitchell order of κ in M ”; more precisely, the ordertype of the collection of $\nu \geq \kappa^{+M}$ which index an extender with critical point κ . On the other hand $o^M(\kappa)$ denotes the least primitively recursively closed ordinal which does not index an extender with critical point κ .

Theorems 45, 46, and 47 are proved in sections 5.1, 5.2, and 5.3, respectively.

5.1. Proof of Theorem 45. In this section we prove Theorem 45. Let $\kappa := \max(cf(\gamma)^+, \omega_2)$, and pick a large regular $\chi \gg \gamma^{+K}$.¹⁴ Let S be the set of all weakly internally approachable $X \in P_\kappa(H_\chi)$ such that $X \cap \text{Ord}$ is closed under limits of cofinality $< cf(\gamma)$ (and under limits of cofinality ω , if $cf(\gamma) = \omega$) and $D \subset X$. S is stationary in $P_\kappa(H_\chi)$ by a standard closure argument.

For each $X \in S$, use the same notation for the coiteration data as in Lemma (39) (i.e. $\pi_X, H_X, K_X, \theta_X, \langle N_\iota^X, \kappa_\iota^X, \nu_\iota^X, \delta_\iota^X, \tau_\iota^X \mid \iota \leq \theta_X \rangle, \dots$). If $r \in \text{range}(\pi_X)$ we will denote $\pi_X^{-1}(r)$ by r_X . Let $Q_X := K_X \restriction \gamma_X$.

By Lemma 39, for almost every $X \in S$ we have:

- (1) $\alpha_X^{+K_X}$ is not a cardinal in K . Then by the convention in Definition 9, if K_X is an initial segment of K then κ_0^X is defined to be $\alpha_X, \tau_0^X = \alpha_X^{+K_X}, \delta_0^X < \alpha_X^{+K}$ is maximal such that τ_0^X is a cardinal in $K \restriction \delta_0^X$, and $(N_0^X)^* = K \restriction \delta_0^X$.
- (2) If ν_0^X is defined (i.e. K_X is not an initial segment of K) then $\nu_0^X < \alpha_X^{+K}$.
- (3) The K side truncates by stage 1. By Lemma 39 there are only two possibilities:
 - *CASE 0*: $\tau_0^X \geq \alpha_X$ and K truncates at stage 0, or
 - *CASE 1*: $\tau_0^X < \alpha_X$ and $(\nu_0^X)^{+K_X}$ is not a cardinal in $K_1 := \text{ult}(K, E_{\nu_0^X})$. By the convention in Definition 9, the objects $\kappa_1^X, \tau_1^X, \delta_1^X$, and $(N_1^X)^*$ are defined even if K_X is an initial segment of K_1 (if that happens here, then $\kappa_1^X = \nu_0^X$).
- (4) The K_X side of the coiteration is trivial; note that this is equivalent to $\nu_i^X = o^{K_X}(\kappa_i^X)$ whenever ν_i^X is defined.
- (5) Let $\iota_X = 0$ if Case 0 holds and $\iota_X = 1$ if Case 1 holds (i.e. ι_X is the stage of the first truncation). Then for every $i \geq \iota_X$: the ultimate projectum of $(N_i^X)^*$ is $\leq \kappa_i^X$ and $(N_i^X)^*$ is sound above κ_i^X .

So if Case 0 holds then $(N_0^X)^*$ is defined (even if K_X is an initial segment of K) and $(N_0^X)^*$ projects to or below α_X ; in particular it projects strictly below γ_X . If Case 1 holds, then $(N_1^X)^*$ is defined (even if K_X is an initial segment of K_1) and $(N_1^X)^*$ projects to ν_0^X ; the next claim will then imply that $(N_1^X)^*$ projects strictly below γ_X :

Claim 49. *If Case 1 holds, then $\nu_0^X < \gamma_X$.*

¹⁴For the purposes of Theorem 45, $\chi > \gamma^{+K}$ suffices.

Proof. Suppose to the contrary that $\nu_0^X \geq \gamma_X$. We already know by item 2 in the list above that $\nu_0^X < \alpha_X^{+K}$, so $\gamma_X \in (\alpha_X, \alpha_X^{+K})$; in particular γ_X is singular in K . Let δ_X be maximal such that γ_X is regular in $\widehat{K||\delta_X}$. Then the ultimate projectum of $\widehat{K||\delta_X}$ is $\leq \gamma_X$ and γ_X is definably singularized over $\widehat{K||\delta_X}$. Since $\nu_0^X \geq \gamma_X$, then $K_X|_{\gamma_X}$ is an initial segment of K , so it makes sense to form the canonical lifting of $\sigma_X \upharpoonright \gamma_X$ to $\widehat{K||\delta_X}$. By Lemma 43, this canonical lifting is wellfounded for almost all $X \in S$; let $\tilde{\sigma}_X : \widehat{K||\delta_X} \rightarrow P_X$ be one such lifting. Now γ_X is regular in $\widehat{K||\delta_X}$, so $\tilde{\sigma}_X(\gamma_X) = \sup \sigma_X[\gamma_X] = \gamma$; and since γ_X is definably singular over $\widehat{K||\delta_X}$, then γ is definably singular over P_X . But $P_X \in K$ by Lemma 43, so γ is singular in K . Contradiction. $\square$

To summarize:

$$(23) \quad \begin{aligned} & \widehat{(N_0^X)^*} \text{ is sound and projects strictly below } \gamma_X \text{ in Case 0} \\ & \widehat{(N_1^X)^*} \text{ is sound and projects strictly below } \gamma_X \text{ in Case 1} \end{aligned}$$

Since the K_X side of the coiteration is trivial and the last iterate on the K side must end-extend K_X , there is some minimal stage Ω_X such that the extender sequences of $N_{\Omega_X}^X$ and K_X agree below γ_X . Define:

$$(24) \quad N_X := \begin{cases} (N_0^X)^* & \text{if } \Omega_X = 0 \\ (N_1^X)^* & \text{if } \Omega_X = 1 \\ N_{\Omega_X}^X & \text{if } \Omega_X > 1 \end{cases}$$

The height of N_X is at least γ_X . Furthermore, since there may be a truncation at state Ω_X , if $\Omega_X > 1$ then:

$$(25) \quad \text{If } \Omega_X > 1 \text{ then } N_X \text{ is not necessarily the same as } (N_{\Omega_X}^X)^*.$$

If $\Omega_X > 0$ (i.e. $K_X|_{\gamma_X}$ is not an initial segment of K) then $J_X := \{j | j \geq \iota_X \text{ and } \nu_j^X < \gamma_X\}$ is nonempty and Ω_X is the strict supremum of J_X . We will later see that the critical points of the coiteration are unbounded in γ_X , so the “strict” here will be superfluous and Ω_X will never be 0 or 1. Furthermore, using (23) it is easy to show:

$$(26) \quad \begin{aligned} & \text{The ultimate projectum of } \widehat{N_X} \text{ is } \leq \sup(\{\nu_j^X | j \in J_X\}) \\ & \text{and } \widehat{N_X} \text{ is sound above } \sup(\{\nu_j^X | j \in J_X\}) \end{aligned}$$

Lemma 50. *For almost every $X \in S$: γ_X is not Σ^* singularized over $\widehat{N_X}$.*

Proof. Suppose for a contradiction that γ_X were definably singularized over $\widehat{N_X}$ for stationarily many $X \in S$. For such X pick η_X maximal such that γ_X is regular in $\widehat{N_X||\eta_X}$. Then the ultimate projectum of

$\widehat{N_X || \eta_X}$ is $\leq \gamma_X$ and $\widehat{N_X || \eta_X}$ is sound above γ_X ; in case $\eta_X = ht(N_X)$ (i.e. γ_X is regular in N_X) this is guaranteed by (26). Then the Frequent Extensions of Embeddings Lemma yields liftings $\tilde{\sigma}_X : \widehat{N_X || \eta_X} \rightarrow P_X$, where $\gamma = \tilde{\sigma}_X(\gamma_X)$ is regular in, but definably singularized over, the structure P_X . By Lemma 43, $P_X \in K$ and thus γ is singular in K , a contradiction. $\square$

Lemma 51. *For almost every $X \in S$, $D_X := \{\kappa_i^X | \pi_{i, \Omega_X}^K(\kappa_i^X) = \gamma_X\}$ is a closed unbounded subset of γ_X . (Note that we do not assume that γ has uncountable cofinality; the following proof would be a bit simpler if that were the case).*

Proof. First, we note that

$$(27) \quad \begin{array}{l} \text{The critical points of the coiteration are cofinal in } \gamma_X \\ \text{for almost every } X \in S \end{array}$$

Equivalently, $\{\nu_\iota^X | \iota \in J_X\}$ is cofinal in γ_X . Suppose not; let $s_X := \sup\{\nu_\iota^X | \iota \in J_X\} (= \sup\{\nu_\iota^X | \iota < \Omega_X\})$. By (26), $\widehat{N_X}$ is sound above s_X ; but then γ_X would be $\Sigma^*(\widehat{N_X})$ -singularized, contradicting Lemma 50.

Now we show D_X is unbounded for almost all $X \in S$; closure then follows by basic properties of directed systems. Suppose for a contradiction that D_X is bounded for stationarily many $X \in S$. We will show this implies γ_X is definably singularized over $\widehat{N_X}$, which again contradicts Lemma 50. For the rest of the proof we omit the X subscripts and superscripts when there is no danger of confusion. Since D_X is bounded below γ_X there is some $i^* < \Omega_X$ such that for every $j \in [i^*, \Omega_X)$, $\gamma_j := \pi_{j, \Omega_X}^{-1}(\gamma_X)$ is strictly greater than κ_j . Also demand that the degree of the critical points (on the K side) past stage i^* is fixed at some $n \in \omega$. This can be arranged since by (27), Ω_X is a limit ordinal and the degrees of the critical points of a normal iteration constitute a nonincreasing sequence of natural numbers. Then (27) also guarantees:

$$(28) \quad \omega \rho_{\widehat{N_X}}^{n+1} < \gamma_X \leq \omega \rho_{\widehat{N_X}}^n$$

The following claim shows that γ_X is definably singularized over $\widehat{N_X}$, thus contradicting Lemma 50 and completing the proof of Lemma 51:

Claim 52. *Let $\tilde{h}_j^{n+1} = \tilde{h}_{\tilde{N}_j}^{n+1}(-, p_{\tilde{N}_j})$. Then $\gamma_X \cap \pi_{i^*, \Omega_X} \circ \tilde{h}_{i^*}^{n+1}[\kappa_{i^*}]$ is a cofinal subset of γ_X . Note that the relation $\pi_{i^*, \Omega_X} \circ \tilde{h}_{i^*}^{n+1}[\kappa_{i^*}]$ is $\Sigma_1^{(n)}(\widehat{N_X})$; so this claim shows γ_X is definably singularized over $\widehat{N_X}$.*

Proof of Claim 52. Recall that $\widehat{N_{i^*}^X}$ is sound above κ_{i^*} ; so we just need to show that π_{i^*, Ω_X} is continuous at γ_{i^*} . This, in turn, requires showing that for every $j \in [i^*, \Omega_X)$, the map $\pi_{j, j+1}$ is continuous at γ_j . By (28), either every such γ_j equals $\omega \rho_{\widehat{N_j}}^n$, or else every γ_j is an element of the n -th reduct of $\widehat{N_j}$. In the former case, continuity of the iteration maps at each γ_j follows from the closeness of the applied extenders and the fact that $n = \text{degree}_{\widehat{N_j}}(\kappa_j)$. In the latter case, notice first that since Lemma 50 guarantees that γ_X is regular in $\widehat{N_X}$, then by elementarity of π_{j, Ω_X} we have γ_j is regular in $\widehat{N_j}$; in particular, $cf^{\widehat{N_j}}(\gamma_j) > \kappa_j$. Since γ_j is an element of the n -th reduct and $n = \text{degree}_{\widehat{N_j^X}}(\kappa_j^X)$, then the $*$ -ultrapower at stage j agrees with the coarse ultrapower on the n -th reduct of $\widehat{N_j}$, and is thus continuous at γ_j since $cf^{\widehat{H_{N_j}^n}}(\gamma_j) > \kappa_j$. $\square$ (Claim 52)

$\square$ (Lemma 51)

Without loss of generality, for the rest of this section assume the statement of Lemma 51 holds for every $X \in S$. Then in particular every Ω_X is a limit ordinal, so for each X there is an $n_X \in \omega$ past which the degrees of the critical points stabilize; i.e., there is some $i_X^* < \Omega_X$ such that for every $j \in [i_X^*, \Omega_X)$, n_X is maximal such that κ_j^X is an element of the n_X -th reduct of $\widehat{N_j^X}$. Without loss of generality assume n_X is the same for every $X \in S$, say $n_X = n$.

Also, γ_X is measurable in $\widehat{N_X}$, since it is the image of measurables via the iteration maps. This implies:

Claim 53. γ does not index an extender in K

Proof. Suppose it did; say γ indexes an extender on some $\mu < \gamma$ in K . Then γ_X indexes an extender on $\mu_X := \sigma_X^{-1}(\mu)$ in K_X . Since K_X and N_X agree below γ_X , this implies that $o^{N_X}(\mu_X) \geq \gamma_X$, so that there are cofinally many extenders below γ_X which index extenders on μ_X in N_X ; this contradicts Lemma 7, since γ_X is measurable in N_X (recall we are assuming 0-pistol does not exist; i.e. no mice have overlapping extenders). $\square$

So the top extender of N_X (if it has one) has critical point $\geq \gamma_X$; let μ_X denote this critical point. Then by Lemma 7 $\text{ult}^*(\widehat{N_X}, \sigma_X \upharpoonright \gamma_X)$ a premouse (i.e. its top extender, if it has one, is weakly amenable), which we'll denote by N'_X , and is in fact an initial segment of K by Lemma 43.

The map $\tilde{\sigma}_X$ is $\Sigma_0^{(n)}$ preserving and n -cofinal, since each $\widehat{N_X}$ is sound above γ_X . Let τ_X denote the successor of γ_X in $\widehat{N_X}$. Let $D'_X := \sigma_X[D_X]$ and $U'_X = \tilde{\sigma}_X(U_X)$, where $U_X := \mathcal{U}^{N_X}(\gamma_X, 0)$, i.e. the measure on γ_X of order zero in $\widehat{N_X}$ (recall the convention 8). Here $\tilde{\sigma}_X(U_X)$ refers to the top predicate of $\widehat{N'_X}$, if U_X is the top predicate of $\widehat{N_X}$.

For every pair $X \subset Y$ in S , let $\sigma_{XY} = \sigma_Y^{-1} \circ \sigma_X \upharpoonright \gamma_X$. Using an interpolation-like argument with the map $\tilde{\sigma}_X$, and the fact that $\widehat{N_X}$ is sound above γ_X , construct the following maps:

- $\tilde{\sigma}_{XY} : \widehat{N_X} \rightarrow_{\Sigma_0^{(n)}} \widehat{N_{XY}}$ which is n -cofinal and extends $\sigma_{XY} \upharpoonright \gamma_X$; the ultrapower really is a premouse by Lemma 7, since γ_X does not index an extender in N_X .
- $\sigma'_{XY} : \widehat{N^{XY}} \rightarrow_{\Sigma_0^{(n)}} \widehat{N'_X}$ which extends $\sigma_Y \upharpoonright \gamma_Y$.¹⁵

The following diagram depicts the situation; the “hats” are omitted.

$$\begin{array}{ccc}
 & N'_X & \\
 \tilde{\sigma}_X \nearrow & & \nwarrow \sigma'_{XY} \\
 N_X & \xrightarrow{\tilde{\sigma}_{XY}} & N_{XY}
 \end{array}$$

$$Q_X \xrightarrow{\sigma_{XY}} Q_Y$$

Also note that $\tilde{\sigma}_X$ and $\tilde{\sigma}_{XY}$ are both continuous at $\gamma_X^{+\widehat{N_X}}$, since they are n -cofinal and $\gamma_X < \omega\rho_{\widehat{N_X}}^n$.

Claim 54. N_{XY} is an initial segment of N_Y .

Proof. The expansions of both N_{XY} and N_Y are sound above γ_Y , and the preservation degree of σ'_{XY} guarantees that N_{XY} is normally iterable above γ_Y . Finally, N_{XY} and N_Y both end-extend $K_Y \upharpoonright \gamma_Y$ and coiterate above γ_Y , since by claim (53) γ_Y does not index an extender in either N_{XY} or N_Y . It follows that one of N_{XY}, N_Y is an initial segment of the other.

If N_Y were a proper initial segment of N_{XY} , then $\gamma_Y^{+\widehat{N_Y}} < \gamma_Y^{+\widehat{N_{XY}}}$ and $\sigma'_{XY} \upharpoonright \widehat{N_Y} : \widehat{N_Y} \rightarrow \sigma'_{XY}(\widehat{N_Y})$ would be a fully elementary map which extends $\sigma_Y \upharpoonright \gamma_Y$. Using the interpolation lemma, it can be shown that σ'_{XY} and $\tilde{\sigma}_Y$ would agree below $\gamma_Y^{+\widehat{N_Y}}$ (their restrictions to $\gamma_Y^{+\widehat{N_Y}}$ would both just be the canonical lifting of $\sigma_Y \upharpoonright$ to $\widehat{N_Y} \upharpoonright \gamma_Y^{+\widehat{N_Y}}$). But then

¹⁵ σ'_{XY} is defined by $[\xi, f]_{\sigma_{XY}} \mapsto \tilde{\sigma}_X(f)(\sigma_Y(\xi))$, where f is a good $\Sigma_1^{(n-1)}(\widehat{N_X})$ function with $\text{dom}(f) < \gamma_X$ and $\xi < \sigma_{XY}(\text{dom}(f))$

$\gamma^{+N'_Y} = \sup(\tilde{\sigma}_Y[\gamma_Y^{+N_Y}]) = \sup(\sigma'_{XY}[\gamma_Y^{+N_Y}]) < \sigma'_{XY}(\gamma_Y^{+N_{XY}}) = \gamma^{+N'_X}$, which is a contradiction. $\square$

Claim 55. *There are sets T and $\langle E_X | X \in T \rangle$ such that:*

- (1) T is a stationary subset of S
- (2) For every $X \in T$:
 - (a) $E_X \subset D_X$, $\text{otp}(E_X) = cf(\gamma)$, and E_X is cofinal in γ_X .
 - (b) If $cf(\gamma) > \omega$ then every element of E_X has cofinality $< cf(\gamma)$ (recall that each X is closed under increasing sequences of length $< cf(\gamma)$; so every element of E_X is mapped continuously by σ_X).
- (3) For every pair $X \subset Y$ in T , $E_Y \subset \text{range}(\sigma_{XY})$

Proof. If $cf(\gamma) = \omega$: for each $X \in S$, pick some cofinal $E_X \subseteq D_X$ of ordertype ω , and let $E'_X = \sigma_X[E_X]$. Since X is weakly internally approachable and $|E'_X| = \omega$, there is some Z_X which is both an element and subset of X such that $E'_X \subset Z_X$. Use the Fodor Lemma for $P_\kappa(H_X)$ to obtain a stationary $T \subseteq S$ such that Z_X is the same for all $X \in T$.

If $cf(\gamma) > \omega$: fix some set B which is club in γ , $\text{otp}(B) = cf(\gamma)$, and every element of B has cofinality $< cf(\gamma)$. Let $T = \{X \in S \mid B \subset X\}$; i.e. those X such that $B \subset \text{range}(\sigma_X)$. Recall that σ_X is continuous on cofinalities $< cf(\gamma)$, which implies that $\sigma_X^{-1}[B]$ is club in γ_X . For $X \in T$ let $E_X = \sigma_X^{-1}[B] \cap D_X$, and $E'_X = \sigma_X[E_X]$.

It is straightforward to check that the sets constructed satisfy the conclusions of the claim. $\square$

For $X \in T$, let $E'_X := \sigma_X[E_X]$.

Claim 56. *For every pair $X \subset Y$ in T , E'_Y is eventually contained in D'_X .*

Proof. We already know by Claim (55) that $E'_Y \subset \text{range}(\sigma_X)$; so to prove the current claim it suffices to show that $\sigma_{XY}^{-1}[E_Y]$ is eventually contained in D_X . Now let β be any element of E_Y such that $\beta > \sigma_{XY}(\min(D_X))$ and $\beta > \eta_{XY}$, where $\eta_{XY} < \gamma_Y$ is the parameter from which $\tilde{h}_{XY}^{n+1}$ is defined over $\widehat{N^Y}$ (i.e. $\tilde{h}_{XY}^{n+1} = \tilde{h}_Y^{n+1}(\eta_{XY})$; if $N_{XY} = N^Y$ then just let $\eta_{XY} = 0$); note this makes sense because N_{XY} is an initial segment of N_Y . Then $\bar{\beta} := \sigma_{XY}^{-1}(\beta) \in D_X$. If not, then by Lemma 24, there is some $\bar{\xi} < \bar{\beta}$ such that $\tilde{h}_X^{n+1}(\bar{\xi}) \in [\bar{\beta}, \gamma_X)$. Then since $\tilde{\sigma}_{XY}$ is $\Sigma_1^{(n)}$ preserving, $\tilde{h}_{XY}^{n+1}(\bar{\xi}) = \tilde{h}_Y^{n+1}(\eta_{XY})(\bar{\xi}) \in [\beta, \gamma_Y)$ where $\xi := \sigma_{XY}(\bar{\xi})$; so $\prec \xi, \eta_{XY} \succ \beta$ and therefore $\tilde{h}_Y^{n+1}[\beta] \cap [\beta, \gamma_Y) \neq \emptyset$. This contradicts Lemma 24 and the fact that $\beta \in D_Y$. $\square$

Claim 57. $U'_X \subseteq U'_Y$ for every pair $X \subset Y$ in T .

Proof. CASE 1: There are cofinally many $\kappa \in E'_Y$ which are *not* measurable in K .

Let F'_Y be a cofinal subset of such points in E'_Y . By the last claim, we can also assume that for every $\lambda \in F'_Y$, $\lambda_X := \sigma_X^{-1}(\lambda) \in D_X$. By elementarity of σ_X , for every $\lambda \in F'_Y$, λ_X is not measurable in K_X and $\sigma_Y^{-1}(\lambda)$ is not measurable in K_Y . This means that at the stage in K vs. K_X where λ_X was the critical point, the extender applied on the K side was the one of order zero (since K_X is not moved in the coiteration), and similarly for the K vs. K_Y coiteration. So:

- (1) $\sigma_X^{-1}[F'_Y]$ is a generating sequence for U_X ; i.e. $z \in U_X$ iff $\sigma_X^{-1}[F'_Y]$ is eventually contained in z .
- (2) $\sigma_Y^{-1}[F'_Y]$ is a generating sequence for U_Y ; i.e. $z \in U_Y$ iff $\sigma_Y^{-1}[F'_Y]$ is eventually contained in z .

Item (1) implies that F'_Y is a generating sequence for U'_X , and item (2) implies that it is a generating sequence for U'_Y , so $U'_X \subseteq U'_Y$. To see that F'_Y is a generating sequence for U'_X (the proof that it is a generating sequence for U'_Y is similar): notice that $\tilde{\sigma}_X$ is n -cofinal and $\tau_X \leq \omega \rho_{N_X}^n$, so $\tilde{\sigma}_X$ is continuous at τ_X . Then for any $z \in U'_X$, there is some $\delta < \tau_X$ such that $z \in \tilde{\sigma}(J_\delta^{E^{N_X}})$. Since U_X is weakly amenable with respect to N_X (this is one of the requirements to be on a premouse extender sequence), $U_X \cap J_\delta^{E^{N_X}}$ is an element of $\widehat{N_X}$; let $\mathcal{F} : \gamma_X \rightarrow_{\text{onto}} U_X \cap J_\delta^{E^{N_X}}$ be an enumeration in N_X . Then $\Delta\mathcal{F} \in U_X$, so it contains a tail end of $\sigma_X^{-1}[F'_Y]$. So $\tilde{\sigma}(\Delta\mathcal{F}) = \Delta(\tilde{\sigma}(\mathcal{F}))$ contains a tail end of F'_Y (recall $F'_Y \subseteq E'_Y \subset \text{range}(\sigma_X)$), say above λ_0 . Also, since $z \in U'_X \cap \tilde{\sigma}(J_\delta^{E^{N_X}})$ then it is on the $\tilde{\sigma}(\mathcal{F})$ enumeration, so $\Delta\tilde{\sigma}(\mathcal{F})$ is eventually contained in z ; say this happens at all points above λ_1 . Then z contains all points in F'_Y above $\max(\lambda_0, \lambda_1)$.

CASE 2: Cofinally many $\lambda \in E'_Y$ are measurable in K . The proof is similar to case 1. Let F'_Y be a cofinal sequence of such points in E'_Y , and again we can assume by the last claim that for every $\lambda \in F'_Y$, $\lambda_X := \sigma_X^{-1}(\lambda)$ is a critical point in the K vs. K_X coiteration. For every $\lambda \in F'_Y$, λ_X is measurable in K_X and $\lambda_Y := \sigma_Y^{-1}(\lambda)$ is measurable in K_Y . So at the stage in K vs. K_X where λ_X was the critical point, the extender applied on the K side had order > 0 (since the K_X side did not move), and similarly for the K vs. K_Y coiteration. In particular, at such stages the K side and the K_X side have the same extender of order zero on the critical point. So:

- (1) $\sigma_X^{-1}[F'_Y]$ generates U_X in the following sense: $z \in U_X$ iff for all sufficiently large $\lambda \in F'_Y$, $z \cap \lambda \in \mathcal{U}^{N_X}(\lambda, 0) = \mathcal{U}^{K_X}(\lambda, 0)$.

- (2) $\sigma_Y^{-1}[F'_Y]$ generates U_Y in the following sense: $z \in U_Y$ iff for all sufficiently large $\lambda \in F'_Y$, $z \cap \lambda \in \mathcal{U}^{N_Y}(\lambda, 0) = \mathcal{U}^{K_Y}(\lambda, 0)$.

Item (1) implies that F'_Y generates U'_X in the analogous way; i.e. $z \in U'_X$ iff for sufficiently large $\lambda \in F'_Y$, $z \cap \lambda \in \mathcal{U}^{N'_X}(\lambda, 0) = \mathcal{U}^K(\lambda, 0)$. F'_Y generates U'_Y in the analogous way. The proof of these is similar to case 1. So then $U'_X \subseteq U'_Y$. $\square$

Recall that (the extender generated by) U'_X is on N'_X 's extender sequence, and so is normal and weakly amenable with respect to N'_X . Also, N'_X is an initial segment of K . Using these facts with Claim (57), it is straightforward to see that $U := \bigcup_{X \in T} U'_X$ is an ultrafilter on $P^K(\gamma)$ which is normal and weakly amenable with respect to K and concentrates on non-measurables. By Lemma 42 (the Weak Covering Lemma) γ^{+K} has uncountable cofinality. So by Lemma 30 and Corollary 29, $ult(K, U)$ is wellfounded and U generates K 's extender of order zero.

5.2. Proof of Theorem 46.

5.2.1. *Proof of part 1 of Theorem 46.* Suppose for a contradiction that the first conclusion of Theorem 46 fails, and let $\beta < cf(\gamma)$ be the least ordinal where it fails. Then $S_\beta := \{\lambda < \gamma \mid o_*^K(\lambda) = \beta\}$ is stationary in γ . We will obtain a contradiction by constructing a K -correct measure of order β on some element of S_β .

The proof is similar to the previous section; in particular, all lemmas and claims from that section hold here, since we are now proving a special case of Theorem 45. Let $S_X^\beta := \sigma_X^{-1}[S_\beta]$; then S_X^β is stationary in γ_X since σ_X is continuous on cofinalities $< cf(\gamma)$ and S_β is stationary in γ . We make the following additional requirement on the sets $\langle E_X \mid X \in T \rangle$ constructed in Claim 55:

$$(29) \quad E_X \subset D_X \cap S_X^\beta$$

This is possible because S_β is stationary in γ (in particular, the set B from the proof of Claim 55 can be taken as a subset of S_β in the present case).

From now on assume every $X \in S$ has all the nice properties from the last section (e.g. K_X is not moved in its coiteration with K , D_X is club in γ_X , etc.). For those $X \in S$ with $\beta \in X$, let $S_\beta^X = \sigma_X^{-1}(S_\beta)$. Since σ_X is continuous on cofinalities $< cf(\gamma)$, then S_β^X is stationary in γ_X from V 's perspective and so $S_\beta^X \cap D_X$ is stationary in γ_X . $S_\beta^X \cap D_X$ consists of critical points of the coiteration K vs. K_X which lie on a common thread, and whose order in K_X is exactly β . Furthermore, $\beta < \gamma$ and D_X is a thread of critical points to γ_X , so eventually the critical points

of the coiteration are strictly above $\beta_X := \sigma_X^{-1}(\beta)$ ($= \beta$; though this fact isn't used). Thus if κ_i^X, κ_j^X are two sufficiently large critical points in $S_\beta^X \cap D_X$ with $0 < i < j$, then $\pi_{i,j}^X(\nu_i^X) = \nu_j^X$ since ν_i^X, ν_j^X must pick out the β_X -th extenders on κ_i^X and κ_j^X in N_i^X, N_j^X respectively (recall that since K_X is not moved, then ν_i^X is simply $o^{K_X}(\kappa_i^X)$ for every stage i). This implies:

(30) Whenever κ_i^X is an element and limit of $D_X \cap S_\beta^X$,
the set $D_X \cap S_\beta^X \cap \kappa_i^X$ is a generating sequence for
 $\mathcal{U}^{N_i^X}(\kappa_i^X, \beta)$.

For each $X \in T$, select some λ_X which is an element and limit of the stationary set E_X ; and say λ_X is the critical point used at stage $\bar{\Omega}_X$ (i.e. $\lambda_X = \kappa_{\bar{\Omega}_X}^X$). Also, WLOG assume there are no truncations for stages $j \in [\bar{\Omega}_X, \Omega_X)$ (this is possible because Ω_X is a limit ordinal). Let $\bar{N}_X := N_{\bar{\Omega}_X}^X$.¹⁶ Note that $o_*^{\bar{N}_X}(\lambda_X) > \beta_X = o_*^{K_X}(\lambda_X)$ and $D_X \cap S_\beta^X \cap \lambda_X$ is a generating sequence for $\mathcal{U}^{\bar{N}_X}(\lambda_X, \beta)$, by (30).

Apply Fodor's Lemma to find a stationary $T' \subset T$ such that $\sigma_X(\lambda_X)$ is the same for every $X \in T'$; call this common object $\bar{\lambda}$. Now σ_X maps λ_X cofinally to $\bar{\lambda}$, since $\lambda_X \in E_X$ and σ_X is continuous on cofinalities $< cf(\gamma)$. Also, λ_X is measurable in $\bar{N}_X$ and so $o^{\bar{N}_X}(\mu) < \lambda_X$ for every $\mu < \lambda_X$. So by Lemma 43, for almost every $X \in T'$ $\widehat{\bar{N}_X'} = \widehat{ult^*(\bar{N}_X, \sigma_X \upharpoonright \lambda_X)}$ is an initial segment of K . Let $\bar{U}_X$ denote $\mathcal{U}^{\bar{N}_X}(\lambda_X, \beta_X)$ and $\bar{U}_X'$ denote the image of $\bar{U}_X$ via $\tilde{\sigma}_X$. By elementarity of $\tilde{\sigma}_X$ and the fact that $\beta_X < \lambda_X$, $\bar{U}_X'$ has Mitchell order β in $\bar{N}_X'$. Also, E_X' is a generating sequence for it; the proof is similar to Case 1 of Claim 57.

Next we show that whenever $X \subset Y$ are both in T' , $\bar{U}_X' \subset \bar{U}_Y'$. Similarly to the proof of Theorem 45, using an interpolation-like argument we construct a map $\tilde{\sigma}_{XY} : \widehat{\bar{N}_X} \rightarrow_{\Sigma_0^{(n)}} \widehat{\bar{N}_{XY}}$ which extends $\sigma_{XY} \upharpoonright \lambda_X$ (note this latter map is cofinal into λ_Y), and argue that $\bar{N}_{XY}$ is an

¹⁶Unlike the scenario in section 45, it is always the case that $N_{\bar{\Omega}_X}^X = (N_{\bar{\Omega}_X}^X)^*$ since we chose $\bar{\Omega}_X$ to be a stage past all truncations. This allows a construction of the K -extender to be a bit more direct than what is presented in this paper. In particular, instead of lifting $\sigma_X \upharpoonright \lambda_X$ to $N_{\bar{\Omega}_X}^X$ as we do below, one could instead lift $\sigma_X \upharpoonright (\nu_{\bar{\Omega}_X}^X)^{+K_X}$ to a map $\tilde{\sigma}_X$ with domain $N_{\bar{\Omega}_X+1}^X$; this is possible in part because since there is no truncation at stage $\bar{\Omega}_X + 1$, then K_X 's cardinal successor of $\nu_{\bar{\Omega}_X}^X$ is also cardinal in $N_{\bar{\Omega}_X+1}^X$. Then for each $X \in T'$, one could define a map $F_X : P^K(\bar{\lambda}) \cap \text{range}(\sigma_X) \rightarrow P(o^K(\bar{\lambda}))$ by $z \mapsto \tilde{\sigma}_X \circ \bar{F}_X \circ \sigma_X^{-1}(z)$, where $\bar{F}_X = E_{\nu_{\bar{\Omega}_X}^X}^{N_{\bar{\Omega}_X}^X}$. Then we would argue that the F_X cohere with each other, and $\bigcup_{X \in T'} F_X$ would be a $(\bar{\lambda}, o^K(\bar{\lambda}))$ -extender which is on K 's extender sequence, which is a contradiction.

initial segment of $\bar{N}_Y$ (note $\bar{N}_{XY}$ and $\bar{N}_Y$ coiterate above λ_Y , as this ordinal is measurable in both and thus does not index an extender in either). Also, similarly to Claim 56 we have:

For every pair $X \subset Y$ in T' , $\sigma_{XY}^{-1}[E_Y \cap \lambda_Y]$ is a cofinal subset of λ_X which is eventually contained in $D_X \cap \lambda_X$.

Since $E_Y \subset S_\beta^Y \cap \text{range}(\sigma_{XY})$, (5.2.1) implies that:

$$(31) \quad \sigma_{XY}^{-1}[E_Y \cap \lambda_Y] \text{ is eventually contained in } D_X \cap S_\beta^X \cap \lambda_X$$

Recall from (30) that $D_X \cap S_\beta^X \cap \lambda_X$ is a generating sequence for $\bar{U}_X$ (the measure on λ_X of order β_X in $\bar{N}_X$), and also that $E_Y \cap \lambda_Y$ is a generating sequence for $\bar{U}_Y$ (because $E_Y \cap \lambda_Y$ is a cofinal subset of $D_Y \cap S_\beta^Y \cap \lambda_Y$). Recall also that the pointwise image of a generating sequence for $\bar{U}_X$ is a generating sequence for $\bar{U}'_X$ (and similarly for U_Y of course). These facts, together with (31), imply that for every pair $X \subset Y$ in T' , there is a generating sequence of $\bar{U}'_Y$ —namely E'_Y —which is eventually contained in the generating sequence $\sigma_X[D_X \cap S_\beta^X \cap \lambda_X]$ of $\bar{U}'_X$. So $\bar{U}'_X \subset \bar{U}'_Y$.

As in the proof of Theorem 45, $\bar{U} := \bigcup_{X \in T} \bar{U}'_X$ is a normal, weakly amenable K -ultrafilter over $\bar{\lambda}$. By Lemma 30, $\text{ult}(K, U)$ is wellfounded; let $j : K \rightarrow_U K'$. Since each $\bar{U}'_X$ generated $\bar{N}'_X$'s (coherent) extender of Mitchell order β , then it is easy to see that $o_*^{K'}(\bar{\lambda}) = \beta$. Then by Corollary 29, $\bar{U}$ generates $E_{o_{K'}(\bar{\lambda})}^K$ which has Mitchell order β ; so $o_*^K(\bar{\lambda}) > \beta$, a contradiction.

5.2.2. Proof of part 2 of Theorem 46. For part 2, we need to show that for every $\beta < cf^V(\gamma)$, K has a normal measure on γ which concentrates on $a_\beta := \{\xi < \gamma \mid o_*^K(\xi) = \beta\}$. Let S_{a_β} be the collection of $\lambda < \gamma$ such that K has a normal measure on λ concentrating on $a_\beta \cap \lambda$; by part 1 of Theorem 46, S_{a_β} contains a club (in V). By Theorem 47 (proved below), K has a normal measure on γ which concentrates on a_β ; i.e. the Mitchell order of this measure is β , which finishes the proof of Theorem 46, modulo the proof of Theorem 47.

5.3. Proof of Theorem 47. Let a and S_a be as in the hypothesis of Theorem 47; i.e. $a \in P^K(\gamma)$, S_a is a V -stationary subset of γ , and for every $\lambda \in S_a$ there is a normal K -measure concentrating on $a \cap \lambda$. For each $\lambda \in S_a$, pick a normal measure $\mathcal{W}_\lambda \in K$ which concentrates on $a \cap \lambda$. By Corollary 29 $\mathcal{W}_\lambda$ generates an extender on K 's extenders sequence, say the extender indexed by μ_λ ; i.e. $\mathcal{W}_\lambda = E_{\mu_\lambda, \lambda}^K$. For simplicity, choose $\mathcal{W}_\lambda$ to be the minimal measure concentrating on $a \cap \lambda$.

As in Lemma 51, for every $X \in S$ there is a closed unbounded set $D_X \subset \gamma_X$ consisting of critical points of the coiteration which all lie on a common thread to γ_X . Use the same notation as section 5.1; in particular, $N_X := N_{\Omega_X}^X$. Let $S_a^X := \sigma_X^{-1}[S]$; since σ_X is continuous on $\text{cof}(\omega)$ then S_a^X is stationary in γ_X from V 's perspective. If $\kappa_i^X \in D_X \cap S_a^X$, then elementarity of σ_X implies that K_X has a measure on κ_i^X which concentrates on $a_X := \sigma_X^{-1}(a)$, and the least such measure (in K_X 's Mitchell order) generates the extender indexed by $\mu_i^X := \sigma_X^{-1}(\mu_{\sigma_X(\kappa_i^X)})$. In particular, $\mu_i^X < o^{K_X}(\kappa_i^X) = \nu_i^X$ (recall the last equality is because the K_X side does not move), so:

$$(32) \quad \text{For every } \kappa_i^X \in S_a^X \cap D_X: E_{\mu_i^X}^{K_X} = E_{\mu_i^X}^{N_X} = E_{\mu_i^X}^{N_X}.$$

Let $\mathcal{W}_i^X$ denote the measure from (32). Then (32) and the elementarity of the iteration maps implies that:

There is a normal measure $\mathcal{W}_X$ such that:

$$(33) \quad \begin{aligned} & \bullet \text{ } cr(\mathcal{W}_X) = \gamma_X \text{ and } \mathcal{W}_X \text{ generates an extender} \\ & \quad \text{on } N_X \text{'s extender sequence;} \\ & \bullet \text{ } a_X \in \mathcal{W}_X; \\ & \bullet \text{ For every } z \in P^{N_X}(\gamma_X): z \in \mathcal{W}_X \text{ iff } z \cap \kappa_i^X \in \\ & \quad \mathcal{W}_i^X \text{ for sufficiently large } \kappa_i^X \text{ in the stationary} \\ & \quad \text{set } D_X \cap S_a^X. \end{aligned}$$

Define a filter $\mathcal{W}$ on $P^K(\gamma)$ as follows:

$$(34) \quad z \in \mathcal{W} \text{ iff } \sigma_X^{-1}(z) \in \mathcal{W}_X \text{ for some } X \in S.$$

We prove that $\mathcal{W}$ is a K -ultrafilter, it is K -correct, and that its Mitchell order is exactly β . Clearly, $a \in \mathcal{W}$ since $a_X \in \mathcal{W}_X$ for every $X \in S$.

Claim 58. $\mathcal{W}$ is a K -ultrafilter.

Proof. It is clear from the definition of $\mathcal{W}$ that for every $z \in P^K(\gamma)$ at least one of $z, \gamma - z$ is in $\mathcal{W}$. It remains to show that $\mathcal{W}$ is proper. Suppose to the contrary that for some $z \in P^K(\gamma)$ there is a pair $X, X' \in S$ such that $z_X \in \mathcal{W}_X$ but $(\gamma_{X'} - z_{X'}) \in \mathcal{W}_{X'}$, where $z_X := \sigma_X^{-1}(z)$ and $z_{X'} := \sigma_{X'}^{-1}(z)$. By (33), this implies:

- (1) $z_X \cap \kappa_i^X \in \mathcal{W}_i^X$ for sufficiently large $\kappa_i^X \in D_X \cap S_a^X$.
- (2) $(\kappa_j^{X'} - z_{X'}) \in \mathcal{W}_j^{X'}$ for sufficiently large $\kappa_j^{X'} \in D_{X'} \cap S_a^{X'}$.

Now select a sufficiently large element κ of the V -stationary set $\sigma_X[D_X] \cap S_a \cap \sigma_{X'}[D_{X'}]$. Item (1) and elementarity of σ_X imply that $z \cap \kappa \in E_{\mu_{\kappa}, \kappa}^K$. But item (2) and elementarity of $\sigma_{X'}$ imply that $(\kappa - z) \in E_{\mu_{\kappa}, \kappa}^K$. This contradicts that $E_{\mu_{\kappa}, \kappa}^K$ is an ultrafilter. $\square$

Claim 59. $\mathcal{W}$ is normal with respect to K ; i.e. whenever $f \in K$, $\text{dom}(f) = \gamma$ and $\text{rng}(f) \subset \mathcal{W}$ then $\Delta f \in \mathcal{W}$.

Proof. Let $\vec{A} \in K$ be a γ -sequence of sets in $\mathcal{W}$. Pick any $X \in S$ with $\vec{A} \in X$. Then $\vec{A}_X := \sigma_X^{-1}(\vec{A}) \in K_X$ is a γ_X -sequence of subsets of γ_X , and $\vec{A}_X \in N_X$ since $P^{K_X}(\gamma_X) \subseteq P^{N_X}(\gamma_X)$ (the latter is because K_X does not move in the coiteration and the least disagreement (if there is one) between N_X and K_X has critical point $\geq \gamma_X$). By the definition of $\mathcal{W}$ and the fact that every set in $\vec{A}$ is in $\mathcal{W}$, then every set on the sequence $\vec{A}_X$ is in the measure $\mathcal{W}_X$. Since $\mathcal{W}_X$ is normal with respect to N_X (recall it generates an extender on N_X 's extender sequence; see (33)), $\Delta \vec{A}_X = \sigma_X^{-1}(\Delta \vec{A}) \in \mathcal{W}_X$. So $\Delta \vec{A} \in \mathcal{W}$ by the definition of $\mathcal{W}$. $\square$

Claim 60. $\text{ult}(K, \mathcal{W})$ is wellfounded

Proof. We already know by Theorem 45 that γ is measurable in K and so γ does not index an extender in K . And by Lemma 42 (Weak Covering Lemma) $\tau := \gamma^{+K}$ has uncountable cofinality. So by Corollary 19 it suffices to show that $\text{ult}(K|\tau, \mathcal{W})$ is wellfounded.

Suppose to the contrary, and let $\{f_n | n \in \omega\} \subset K|\tau$ witness the illfoundedness of $\text{ult}(K|\tau, \mathcal{W})$. Since γ is the largest cardinal in $K|\tau$, then WLOG we can assume $\text{range}(f_n) \subset \gamma$ for every $n \in \omega$. Pick an $X \in S$ with $\{f_n | n \in \omega\} \in X$. Let $\tau_X := \sigma_X^{-1}(\tau)$. Then $\{\sigma_X^{-1}(f_n) | n \in \omega\}$ witnesses that $\text{ult}(K_X|\tau_X, \mathcal{W}_X)$ is illfounded; but this is a contradiction, since $K_X|\tau_X$ is an initial segment of the mouse N_X and $\mathcal{W}_X$ generates an extender on $N_{\Omega_X}^X$'s extender sequence. $\square$

Finally, we note that by lifting stationary sets the measure $\mathcal{W}$ can be equivalently defined as follows:

$$(35) \quad z \in \mathcal{W} \text{ iff } z \cap \lambda \in E_{\mu_\kappa, \kappa}^K \text{ for all but } V\text{-nonstationarily many } \kappa < \gamma$$

6. AN APPLICATION TO STATIONARY SET REFLECTION

6.1. Reflection of a single stationary set at a small cofinality.

If κ is an ordinal of uncountable cofinality, $S \subseteq \kappa$ is stationary, and $\gamma < \kappa$ has uncountable cofinality, we say S *reflects at* γ if $S \cap \gamma$ is stationary in γ . Starting from a Mahlo cardinal, Harrington and Shelah in [4] obtained a model of “Every stationary subset of $\omega_2 \cap \text{cof}(\omega)$ reflects.” In the other direction, if every stationary subset of $\omega_2 \cap \text{cof}(\omega)$ reflects, then Jensen’s global square sequence for L can be used to show that ω_2 is Mahlo in L . Reflection for stationary subsets of a successor cardinal $\geq \omega_3$ are similarly equiconsistent with a Mahlo cardinal. However, requiring the reflection points to have small cofinality yields larger cardinals in K :

Theorem 61. *Assume every stationary subset of S_0^3 reflects at a point of cofinality ω_1 . For $f, g \in {}^{\omega_3}V$, define $f <^* g$ iff $f(\xi) < g(\xi)$ for almost every $\xi \in \omega_3 \cap \text{cof}(< \omega_2)$. Then $h_\nu <^* o_*^K$ for each $\nu < \omega_3^{+K}$.¹⁸*

First, the hypothesis of Theorem (61) is equivalent to requiring that S reflect *stationarily often* in S_1^3 : Suppose $S \subset S_0^3$ is stationary and reflects only nonstationarily often in $\text{cof}(\omega_1)$. Let $D \subset \omega_3$ be an unbounded subset of ω_3 which is closed under ω_1 -limits, and such that for every $\gamma \in D$, $S \cap \gamma$ is nonstationary. Then $S \cap \text{Lim}(D)$ is a stationary subset of S_0^3 which reflects *nowhere* in $\text{cof}(\omega_1)$. So:

(36) Every stationary subset of S_0^3 reflects stationarily often in $\text{cof}(\omega_1)$.

Claim 62. *Almost every $\gamma \in S_1^3$ is regular in K .*

Proof. We will use K ’s “global square” sequence. For the K in the current paper (i.e. for one strong cardinal) the construction in Zeman [11] suffices; see Schimmerling and Zeman [8] and Zeman [12] for much stronger versions. The global square sequence is a sequence $\langle c_\alpha \mid \alpha \in \text{Sing}(K) \rangle$, where $\text{Sing}(K)$ is the collection of all *ordinals* which are singular in K . This sequence has the following properties:

- (1) For every $\alpha \in \text{Sing}(K)$: $c_\alpha \subset \text{Sing}(K)$, $\text{otp}(c_\alpha) < \alpha$, and c_α is closed and unbounded in α .
- (2) If α is singular in K and β is a limit of c_α , then $c_\beta = c_\alpha \cap \beta$.

¹⁸Recall from Definition 6 that for mice below 0-pistol, $o_*^M(\kappa)$ denotes the “Mitchell order of κ in M ”; more precisely, the ordertype of the collection of $\nu \geq \kappa^{+M}$ which index an extender with critical point κ . On the other hand $o^M(\kappa)$ denotes the least primitively recursively closed ordinal which does not index an extender with critical point κ .

Suppose for a contradiction that $T := \{\gamma \in S_1^3 \mid \gamma \in \text{Sing}(K)\}$ is stationary. So for every $\gamma \in T$, c_γ is defined and $c_\gamma \cap \text{cof}^V(\omega)$ is a club in γ . Let $S := \bigcup_{\gamma \in T} \lim(c_\gamma \cap \text{cof}^V(\omega))$; this is a stationary subset of S_0^3 , and for every $\alpha \in S$, α is singular in K and so c_α is defined and has ordertype $< \alpha$. By the Fodor lemma there is a $\bar{\beta} < \omega_3$ and a stationary $\bar{S} \subset S$ such that $\text{otp}(c_\alpha) = \bar{\beta}$ for every $\alpha \in \bar{S}$.

Then $\bar{S}$ has no reflection point in S_1^3 . Suppose $\gamma \in S_1^3$ were a reflection point.

CASE 1: γ is regular in K . Then by Theorem 46, there is (in V) a club $d \subset \gamma$ consisting of K -measurables; but then d cannot intersect $\bar{S}$, since every element of $\bar{S}$ is singular in K .

CASE 2: γ is singular in K . Then c_γ is defined. Then $\bar{S} \cap \lim(c_\gamma)$ has at most one element: suppose $\alpha < \alpha'$ were both in $\bar{S} \cap \lim(c_\gamma)$. Since α' is a limit of c_γ , then $c_{\alpha'} = c_\gamma \cap \alpha'$. Then, since α is a limit of c_γ below α' , it is also a limit of $c_{\alpha'}$ and so $c_\alpha = c_{\alpha'} \cap \alpha$. So $\text{otp}(c_\alpha) < \text{otp}(c_{\alpha'})$, contradicting that α and α' are both in $\bar{S}$. $\square$ (Claim 62)

Let $R_1^3 := S_1^3 \cap \text{Reg}(K)$; by claim 62 this is almost all of S_1^3 . We will first prove a restricted version of Theorem 61 for $\lambda \in \text{cof}(\omega)$:

Claim 63. *Let $\nu < \omega_3^{+K}$ and $b_\nu \in P^K(\omega_3)$ code ν . Then for almost every $\lambda \in S_0^3$: $o_*^K(\lambda) > \text{otp}(b_\nu \cap \lambda)$.*

Proof. First, it is easy to show that the map $\lambda \mapsto \text{otp}(b_\nu \cap \lambda)$ is “the” ν -th canonical function on ω_3 (see section 2). We prove Claim 63 by induction on $\nu < \omega_3^{+W}$. For illustrative purposes, consider the base case $\nu = 0$: suppose $\{\lambda \in S_0^3 \mid o_*^K(\lambda) > 0\}$ did not contain an ω -club, i.e. there is a stationary set T of $\lambda \in S_0^3$ which are not measurable in K . By (36) and Claim 62, T reflects at some $\gamma \in R_1^3$. But by Theorem 46, there is a club of K -measurables in γ , so this club cannot intersect T . Contradiction.

The basic outline of the base case is used for the inductive case. Suppose $\nu > 0$ and the claim holds for all $\tau < \nu$. Let $C_\tau \in V$ be an ω club in ω_3 witnessing the claim for each $\tau < \nu$ (so $o_*^K(\lambda) > \text{otp}(b_\tau \cap \lambda)$ for every $\lambda \in C_\tau$), and fix some sequence $\langle \tau_i \mid i < \text{cf}(\nu) \rangle \in V$ cofinal in ν ; if ν is a successor ordinal we will define its cofinality as 1, and the sequence $\langle \tau_i \mid i < \text{cf}(\nu) \rangle$ will just be $\langle C_{\nu-1} \rangle$. Let $b_\nu \in P^K(\omega_3)$ code ν , and $b_{\tau_i} \in P^K(\omega_3)$ code τ_i . Fix a large regular $\theta > 2^{\omega_3}$ and consider the collection $\tilde{C}$ of all $Z \in P_{\omega_3}(H_\theta)$ such that $Z \prec (H_\theta, \langle (\tau_i, C_{\tau_i}, b_{\tau_i}) \mid i < \text{cf}(\nu) \rangle, \dots)$. For $Z \in \tilde{C}$ let $\pi_Z : H_Z \rightarrow H_\theta$ be the inverse of the Mostowski collapse of Z . Then $\text{cr}(\pi_Z) = Z \cap \omega_3 =: \lambda_Z$ and $\pi_Z^{-1}(\langle b_{\tau_i} \mid i < \text{cf}(\nu) \rangle) = \langle b_{\tau_i} \cap \lambda_Z \mid i < \lambda_Z \rangle$ (if $\text{cf}(\nu) = \omega_3$; similarly if

$cf(\nu) < \omega_3$). It follows that for all $\lambda \in C := \{Z \cap \omega_3 \mid Z \in \tilde{C}\}$:

$$(37) \quad \begin{aligned} otp(b_\nu \cap \lambda) &= \sup_{i < \lambda} (otp(b_{\tau_i} \cap \lambda)) && \text{if } cf(\nu) = \omega_3 \\ otp(b_\nu \cap \lambda) &= \sup_{i < cf(\nu)} (otp(b_{\tau_i} \cap \lambda)) && \text{if } cf(\nu) \leq \omega_2 \\ otp(b_\nu \cap \lambda) &= otp(b_{\nu-1} \cap \lambda) + 1 && \text{if } \nu \text{ successor} \end{aligned}$$

Furthermore, since $Z \prec (H_\theta, \langle C_{\tau_i} \mid i < cf(\nu) \rangle, \dots)$ for $Z \in \tilde{C}$ then λ_Z is a limit of C_{τ_i} for each $i < \lambda_Z$, so:

$$(38) \quad C \cap cof(\omega) \subset \begin{cases} \Delta_{i < cf(\nu)} C_{\tau_i} & \text{if } cf(\nu) = \omega_3 \\ \bigcap_{i < cf(\nu)} C_{\tau_i} & \text{if } cf(\nu) < \omega_3 \\ C_{\nu-1} & \text{if } \nu \text{ successor} \end{cases}$$

Then (37) and (38) imply that $o_*^K(\lambda) \geq otp(b_\nu \cap \lambda)$ for every $\lambda \in C \cap cof(\omega)$. Now suppose for a contradiction that $o_*^K(\lambda) \leq otp(b_\nu \cap \lambda)$ for stationarily many $\lambda \in S_0^3$; let T denote this stationary set. Let $S_\nu := T \cap C$. Then:

$$(39) \quad o_*^K(\lambda) = otp(b_\nu \cap \lambda) \text{ for every } \lambda \in S_\nu.$$

We will obtain a contradiction by finding some $\lambda \in S_\nu$ whose K -order is $> otp(b_\nu \cap \lambda)$.

By (36) and Claim 62:

$$(40) \quad Ref := \{\gamma \in R_1^3 \mid S_\nu \text{ reflects at } \gamma\} \text{ is stationary.}$$

Let S be a stationary set of weakly internally approachable sets in $P_{\omega_2}(H_\chi)$ such that $\gamma'_X := \sup(X \cap \omega_3) \in Ref$ and $X \prec (H_\chi, \in, \dots)$; this is possible since Ref is a stationary subset of S_1^3 .¹⁹ By Lemma 39 we can WLOG assume that for every $X \in S$ the K_X side does not move in its coiteration with K , and the K side truncates by stage 1.

The remainder of the proof is similar to the proof of Theorem 45. The statements of all lemmas and claims through Lemma 24 can be repeated verbatim, and the proofs involve only a minor amendment: in the current proof, for each $X \in S$ the set $\{Y \in S \mid \gamma'_Y = \gamma'_X\}$ might be nonstationary, whereas in the proof of Theorem 45, all $X \in S$ were cofinal in some fixed γ . This difference corresponds to how the parameter γ in the statement of Lemma 31 is interpreted in the proofs: in section 5 we interpreted γ as the ordinal in the hypothesis of Theorem 45, whereas in the current proof we interpret γ as ω_3 .²⁰

¹⁹The reason for setting things up this way—rather than fixing a *single* reflection point γ —will become apparent in the proof in section 6.2. In the current section, we could instead work with a single reflection point γ of S_ν .

²⁰in fact, we could simply use Lemma 44 to argue that, e.g., Lemma 50 holds for *every* $X \in S$.

So for every $X \in S$:

- The critical points of the coiteration of K with K_X are cofinal in $\gamma_X := \sigma_X^{-1}(\omega_3)$ (= the cofinal preimage of γ'_X)
- There is a club D_X in γ_X consisting of critical points of the coiteration of K vs. K_X , which all lie on a common thread to γ_X .

Since each σ_X is continuous on $\text{cof}(\omega)$ and γ'_X is a reflection point of S_ν , then $S_\nu^X := \sigma_X^{-1}[S_\nu \cap \gamma'_X]$ is stationary in γ_X from V 's perspective. Let PRC denote the class of primitive recursively closed ordinals. Now for every sufficiently large²¹ $\kappa_i^X \in D_X \cap S_\nu^X$:

$$\begin{aligned}
 & \text{The Mitchell order of } E_{\nu_i^X}^{N_i^X} = \text{otp}(\text{PRC} \cap [\tau_i^X, \nu_i^X)) \\
 (42) \quad & = \text{otp}(\text{PRC} \cap [\tau_i^X, o^{K_X}(\kappa_i^X))) = o_*^{K_X}(\kappa_i^X) = \\
 & \sigma_X^{-1}(o_*^K(\sigma_X(\kappa_i^X))) = \sigma_X^{-1}(\text{otp}(b_\nu \cap \sigma_X(\kappa_i^X))) = \\
 & \text{otp}(\kappa_i^X \cap b_X)
 \end{aligned}$$

where $b_X := \sigma_X^{-1}(b_\nu)$; also note that if $b_X \in \text{range}(\pi_{i,\Omega_X}^X)$ then $\pi_{i,\Omega_X}^X(b_X \cap \kappa_i^X) = b_X \cap \gamma_X$ since $\kappa_i^X \in D_X$ (here Ω_X is defined as in the discussion around (24)). Now if $\kappa_i^X < \kappa_j^X$ are both in $D_X \cap S_\nu^X$ and $b_X \in \text{range}(\pi_{i,\Omega_X}^X)$ then $\pi_{i,j}^X(b_X \cap \kappa_i^X) = b_X \cap \kappa_j^X$; so (42) implies that $\pi_{i,j}^X(\nu_i^X) = \nu_j^X$. In other words, the extenders applied at stages corresponding to $D_X \cap S_\nu^X$ all lie on a common thread. This implies:

$$\begin{aligned}
 (43) \quad & \text{For every } \kappa_\ell^X \text{ which is an element and limit of the stationary set } D_X \cap S_\nu^X : D_X \cap S_\nu^X \cap \kappa_\ell^X \text{ is a generating sequence for } \mathcal{U}^{N_\ell^X}(\kappa_\ell^X, \text{otp}(b_X \cap \kappa_\ell^X)).
 \end{aligned}$$

Pick some λ_X which is both an element and limit of the stationary set $D_X \cap S_\nu^X$, and using Fodor's Lemma on $P_{\omega_2}(H_\theta)$ we will WLOG assume that $\sigma_X(\lambda_X)$ is fixed at some $\bar{\lambda}$ for every $X \in S$. Let $\bar{\Omega}_X$ denote the stage of the coiteration where λ_X is the critical point, and $\bar{N}_X := N_{\bar{\Omega}_X}^X$; this notation closely follows that of section 5.2.1. Similarly to Lemma 55 there is a stationary $T \subset S$ and a collection of sets $\langle E_X | X \in T \rangle$ such that $E_X \subset D_X \cap S_\nu^X \cap \lambda_X$ and for every $X \subset Y$ in T , $E_Y \subset \text{range}(\sigma_{XY})$. Also, as in section 5.2.1 $\sigma_{XY} \upharpoonright \lambda_X$ can be lifted to a map $\tilde{\sigma}_{XY} : \bar{N}_X \rightarrow \bar{N}_{XY}$ for $X \subset Y$ in T and:

$$(44) \quad \sigma_{XY}^{-1}[E_Y] \text{ is eventually contained in } D_X \cap S_\nu^X \cap \lambda_X.$$

Recall the crucial part was showing $\sigma_{XY}^{-1}[D_Y]$ is eventually contained in D_X ; (44) then follows easily since clearly $\sigma^{-1}[S_\nu^Y] \subseteq S_\nu^X$.

²¹so that there are no truncations at stages in $[i, \Omega_X)$.

Let $\bar{U}_X := \mathcal{U}^{\bar{N}_X}(\lambda_X, otp(b_X \cap \lambda_X))$ and $\bar{U}'_X$ be the image of $\bar{U}_X$ under $\tilde{\sigma}_X$. Similarly to the discussion in section 5.2.1, the pointwise image of $D_X \cap S_\nu^X \cap \lambda_X$ is a generating sequence for $\bar{U}'_X$, and (44) guarantees that $\bar{U}'_X \subset \bar{U}'_Y$ whenever $X \subset Y$ are in T . Then $\bar{U} := \bigcup_{X \in T} \bar{U}'_X$ is a normal, weakly amenable K -ultrafilter on $\bar{\lambda}$, and by Lemma 30 $ult(K, \bar{U})$ is wellfounded; let $j : K \rightarrow_{\bar{U}} K'$. Now $\bar{U}_X$ concentrates on $\{\xi < \lambda_X | o_{*}^{\bar{N}_X}(\xi) = otp(b_X \cap \xi)\}$, so $\bar{U}'_X$ concentrates on $\{\xi < \bar{\lambda} | o_{*}^{N'_X}(\xi) = otp(b_\nu \cap \lambda)\} = \{\xi < \bar{\lambda} | o_{*}^K(\xi) = otp(b_\nu \cap \lambda)\}$ (since N'_X is an initial segment of K). So $o_{*}^{K'}(\bar{\lambda}) = otp(b_\nu \cap \bar{\lambda})$. By Corollary 29, $\bar{U}$ generates the extender $E_{o_{*}^{K'}(\bar{\lambda})}^K$ which has Mitchell order $otp(b_\nu \cap \bar{\lambda})$, so $o_{*}^K(\bar{\lambda}) > otp(b_\nu \cap \bar{\lambda})$, contradicting the fact that $\bar{\lambda} \in S_\nu$. $\square$ (Claim 63)

The rest of Theorem 61 follows from Claim 63 and Theorem 47. Let $\nu < \omega_3^{+K}$ and $b \in P^K(\omega_3)$ code ν . By Claim 63 there is an ω -club $C \subset \omega_3$ such that $o_{*}^K(\lambda) > otp(b \cap \lambda)$ for every $\lambda \in C$; i.e. for every $\lambda \in C$ there is a K -measure on λ which concentrates on $\lambda \cap \{\xi < \omega_3 | o_{*}^K(\xi) = otp(b \cap \xi)\}$. Then for any $\gamma \in S_1^3 \cap Lim(C)$, Theorem 47 guarantees that there is a K -measure on γ which concentrates on $\{\xi < \gamma | o_{*}^K(\xi) = otp(b \cap \xi)\}$. This completes the proof of Theorem 61.

6.2. Simultaneous reflection at a small cofinality. Simultaneous reflection of stationary sets is generally a stronger principle. Magidor [6] proved that the exact consistency strength of “every pair of stationary subsets of $\omega_2 \cap cof(\omega)$ have a common reflection point” is exactly a weakly compact cardinal (the forcing direction was a strengthening of Baumgartner [1]). Again, requiring the reflection points to have small cofinality results in a stronger statement:

Theorem 64. *Assume every pair of stationary subsets of S_0^3 have a common reflection point in S_1^3 . Then there is an inner model of $O(\kappa) = \kappa^+$. In particular, if 0-pistol does not exist then $o_{*}^K(\omega_3) \geq \omega_3^{+K}$.*

First, note that the hypothesis of the theorem is equivalent to:

- (45) Every pair of stationary subsets of S_0^3 have stationarily many common reflection points in $cof(\omega_1)$.

To see (45), suppose $S, T \subset S_0^3$ are stationary, but have only nonstationarily many common reflection points in $cof(\omega_1)$. Let $D \subseteq S_1^3$ be unbounded and closed under limits of cofinality ω_1 , such that S and T have no common reflection points in D . Let $\bar{D}$ be the closure of D ; i.e. $\bar{D} := D \cup Lim(D)$. Let $S' := S \cap \bar{D}$ and $T' := T \cap \bar{D}$. Then S', T' have no common reflection points in $cof(\omega_1)$: let $\gamma \in cof(\omega_1)$ be a limit of both S' and T' . Then γ is an ω_1 -cofinal limit of D , and thus $\gamma \in D$.

So at least one of S, T fails to reflect at γ . Since $S' \subset S$ and $T' \subset T$, then at least one of S', T' fails to reflect at γ .

Fix some $b \in P^K(\omega_3)$ which codes a wellorder of ω_3 ; the goal is to build a K -measure on ω_3 of order $otp(b)$. Using Theorem 61 let C_b be an ω -club in ω_3 such that for every $\lambda \in C_b$, $o_*^K(\lambda) > otp(b \cap \lambda)$. Let S be the collection of $X \in P_{\omega_2}(H_\theta)$ such that $b \in X$, $X \cap \omega_2 \in S_1^2$, and $\gamma'_X := \sup(X \cap \omega_3) \in R_1^3$; note S is stationary (in fact contains a subset which is closed under ω_1 -length chains). Let $\pi_X : H_X \rightarrow H_\theta, N_X, \gamma_X$, etc. be as in the proof of Theorem 61, and let $b_X := \pi_X^{-1}(b)$. For any element κ_i^X of the club $G_X := D_X \cap \pi_X^{-1}(C_b)$: $\nu_i^X = o_*^{K_X}(\kappa_i^X) > otp(b_X \cap \kappa_i^X)$. This implies:

$$(46) \quad \begin{aligned} & \mathcal{U}^{N_X}(\gamma_X, otp(b_X)) \text{ exists, and } z \in \mathcal{U}^{N_X}(\gamma_X, otp(b_X)) \text{ if} \\ & \text{and only if } z \cap \kappa \in \mathcal{U}^{K_X}(\kappa, otp(b_X \cap \kappa)) \text{ for sufficiently} \\ & \text{large } \kappa \text{ in the club } G_X. \end{aligned}$$

Define the filter $\mathcal{F}(\omega_3, b)$ on $P^K(\omega_3)$ by: $z \in \mathcal{F}(\omega_3, b)$ if and only if $z_X \in \mathcal{U}^{N_X}(\gamma_X, otp(b_X))$ for stationarily many $X \in S$ with $z \in X$, where $(z_X, b_X) = \pi_X^{-1}(z, b)$. This makes sense—i.e. $\mathcal{U}^{N_X}(\gamma_X, otp(b_X))$ is defined—by (46).

Claim 65. $\mathcal{F}(\omega_3, b)$ is an ultrafilter on $P^K(\omega_3)$.

Proof. Since S is stationary, $\mathcal{F}(\omega_3, b)$ is clearly a filter; it remains to show that this filter is *proper*. Suppose not; so there is a $z \in P^K(\omega_3)$ such that both $S_z := \{X \in S \mid z \in X \text{ and } z_X \in \mathcal{U}^{N_X}(\gamma_X, b_X)\}$ and $S_{z^c} := \{X \in S \mid z \in X \text{ and } \gamma_X - z_X \in \mathcal{U}^{N_X}(\gamma_X, b_X)\}$ are stationary; also, $\{X \in S \mid z \in X\}$ is the disjoint union of S_z and S_{z^c} . Let $\bar{S}_z := \bigcup_{X \in S_z} \sigma_X[G_X]$ and $\bar{S}_{z^c} := \bigcup_{X \in S_{z^c}} \sigma_X[G_X]$; these are stationary subsets of S_0^3 . Note:

$$(47) \quad \begin{aligned} & \bullet \text{ For every } \lambda \in \bar{S}_z: z \cap \lambda \in \mathcal{U}^K(\lambda, otp(b \cap \lambda)). \\ & \bullet \text{ For every } \lambda \in \bar{S}_{z^c}: \lambda - z \in \mathcal{U}^K(\lambda, otp(b \cap \lambda)). \end{aligned}$$

Let $\bar{R} := \{\sup(X \cap \omega_3) \mid X \in S \text{ and } z \in X\}$; note $\bar{R}$ is almost all of S_1^3 . By (45), $\bar{S}_z$ and $\bar{S}_{z^c}$ have some common reflection point $\bar{\gamma} \in \bar{R}$. Fix a $\bar{X}$ which witnesses that $\bar{\gamma} \in \bar{R}$; i.e. $\bar{X} \in S$ and $z \in \bar{X}$. Now $\mathcal{U}^K(\bar{\gamma}, otp(b \cap \bar{\gamma}))$ exists so either $z \cap \bar{\gamma}$ or $\bar{\gamma} - z$ is in this measure; without loss of generality assume $z \cap \bar{\gamma} \in \mathcal{U}^K(\bar{\gamma}, otp(b \cap \bar{\gamma}))$. By Theorem 47, there is a club $d \subset \bar{\gamma}$ such that for all $\lambda \in d$, $z \cap \lambda \in \mathcal{U}^K(\lambda, otp(b \cap \lambda))$. But d cannot intersect $\bar{S}_{z^c}$, by (47). This contradicts that $\bar{S}_{z^c}$ reflects at $\bar{\gamma}$. $\square$

Claim 66. $\mathcal{F}(\omega_3, b)$ is normal with respect to K ; i.e. if $\vec{A} \in K$ is an ω_3 -sequence of sets from $\mathcal{F}(\omega_3, b)$ then $\Delta \vec{A} \in \mathcal{F}(\omega_3, b)$.

Proof. The proof is the same as that of Claim 59. $\square$

Claim 67. *$ult(K, \mathcal{F}(\omega_3, b))$ is wellfounded.*

Proof. Similar to proof of Claim 60. $\square$

Let $j : K \rightarrow_{\mathcal{F}(\omega_3, b)} K'$; Claim 66 implies that $\omega_3 = cr(j)$. The definition of $\mathcal{F}(\omega_3, b)$ guarantees that it concentrates on $\{\xi < \omega_3 \mid o_*^K(\xi) = otp(b \cap \xi)\}$; so $o_*^{K'}(\omega_3) = otp(b)$. By Corollary 29, $\mathcal{F}(\omega_3, b)$ generates an extender on the K sequence of Mitchell order $otp(b)$. This concludes the proof that $o_*^K(\omega_3) \geq \omega_3^{+K}$.

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E-mail address: `scox@math.uci.edu`

DEPARTMENT OF MATHEMATICS, 340 ROWLAND HALL, UNIVERSITY OF CALIFORNIA, IRVINE, IRVINE, CA 92697-3875, DEPARTMENT PHONE: (949) 824-5503, DEPARTMENT FAX: (949) 824-7993