NONREGULAR ULTRAFILTERS ON ω_2

SEAN COX

ABSTRACT. We obtain lower bounds for the consistency strength of fully nonregular ultrafilters on ω_2.

1. INTRODUCTION

A *nonregular ultrafilter* is a notion which arose from classic questions in model theory about cardinalities of ultrapowers (see Chang and Keisler [1]). Nonregularity is a weakening of countable completeness, and allows for the possibility of a small ultrapower: if an ultrafilter U on ω_1 has the property that $|^{\omega_1}\omega/U| = \omega_1$, then U must be nonregular.

Although in ZFC there is never a countably complete ultrafilter over a cardinal like ω_n, it is consistent relative to large cardinals that there is a nonregular ultrafilter on ω_n ($n \geq 1$). An upper bound for the consistency strength of the existence of a nonregular ultrafilter on ω_1 is ω many Woodin cardinals (see Woodin [13]). Laver [12] showed that if there is an ω_1-dense uniform ideal on ω_1 and $\Diamond$ holds, then there is a nonregular ultrafilter on ω_1. Huberich [9] removed the assumption of $\Diamond$.

The known consistency upper bound for a fully nonregular ultrafilter on ω_2 is higher. Foreman, Magidor, and Shelah [8] obtained such an ultrafilter from an almost huge cardinal. Foreman [7] obtains such an ultrafilter U from a huge cardinal, with many additional properties including the small ultrapower property which appears in part 2 of Theorem 1 below.[1]

Ketonen [11] proved that if there is a nonregular ultrafilter on ω_1, then $0^\sharp$ exists. This was later improved by Donder, Jensen, and Koppelberg [5], and then by Deiser and Donder [6]. Deiser and Donder

2000 *Mathematics Subject Classification.* 03.

This paper is a portion of my PhD thesis ([3]). I would like to thank my advisor, Dr. Martin Zeman, for teaching me set theory, and for all of the support he provided. I would also like to thank Dr. Matt Foreman for helpful discussions on the topic. I gratefully acknowledge support from NSF grant DMS-0500799 (through Martin Zeman), and fellowships from the UCI mathematics department.

[1]Foreman's model actually satisfied $|^{\omega_2}\omega/U| = \omega_1$, which is stronger than the condition appearing in the hypothesis of Theorem 1.

showed that the consistency strength is at least a stationary limit of measurables. In fact, their proof can be slightly modified to show: if there is a nonregular ultrafilter on ω_1 then either 0-sword exists or else

$$\{\nu < \omega_2 | \mathrm{cf}(\nu) = \omega_1 \text{ and } \nu \text{ is measurable in } K\}$$

contains an ω_1-club in V.[2] In either case, there is an inner model M such that V sees an ω_1-club through M's measurables below ω_2.

We build on their argument to obtain lower consistency bounds for fully nonregular ultrafilters on ω_2:

Theorem 1. *Suppose there is a fully nonregular ultrafilter U on ω_2. Then:*

1. *There is an inner model with a cardinal κ of Mitchell order κ^+. In particular, if 0-pistol does not exist, then letting $\kappa = \omega_3^V$, the Mitchell order of κ in the core model K is at least κ^{+K}.*
2. *If $|^{\omega_2}\omega_1/U| = \omega_2$, then there are mice extenders with multiple generators (the consistency strength of this is a bit higher than a measurable cardinal κ with Mitchell order κ^{++}).*

This paper is organized as follows. Section 2 reviews filtrations and canonical functions. Section 3 reviews some facts about nonregular ultrafilters and the related notion of a weakly normal ultrafilter. Section 4 defines the "bounding construction" and proves essential facts about the construction, most of which are abstracted from [6]. Section 5 is the proof of part (1) of Theorem 1, and Section 6 is the proof of part (2) of Theorem 1.

2. Filtrations and Canonical Functions

The notation S^κ_λ will denote the set $\{\alpha < \kappa | \mathrm{cf}(\alpha) = \lambda\}$. We will also write S^m_n instead of $S^{\omega_m}_{\omega_n}$.

As in Deiser and Donder [6] we will make extensive use of canonical functions. If A is a set of cardinality κ, a *filtration of* A is a continuous $\subset$-increasing sequence $\langle A_\alpha | \alpha < \kappa\rangle$ such that $|A_\alpha| < \kappa$ for every $\alpha < \kappa$ and $A = \bigcup_{\alpha<\kappa} A_\alpha$. If κ is regular and uncountable then any two filtrations of A agree on a club. So if $\nu < \kappa^+$ and $\langle A^\nu_\alpha | \alpha < \kappa\rangle$, $\langle B^\nu_\alpha | \alpha < \kappa\rangle$ are filtrations of ν, then the functions $\alpha \mapsto \mathrm{otp}(A^\nu_\alpha)$ and $\alpha \mapsto \mathrm{otp}(B^\nu_\alpha)$ agree on a club. The equivalence class[3] of $\alpha \mapsto \mathrm{otp}(A^\nu_\alpha)$ is called the

[2] 0-sword is an object analogous to 0-sharp; it is an iterable mouse with a top measure of Mitchell order 1. In the absence of 0-sword, the core model K can be constructed and has many properties that L has in the absence of 0-sharp. In this paper we will use 0-pistol instead, which is a stronger object than 0-sword; see Section 4.

[3] modulo the equivalence relation on ${}^\kappa V$ defined by agreement on a club.

ν-th canonical function on κ. There are also inductive characterizations of the canonical functions. Let h_0 be the zero function (on κ); if h_ν is defined let $h_{\nu+1} := h_\nu + 1$; and for limit $\nu < \kappa^+$ fix any cofinal increasing[4] sequence $\langle \nu_\delta | \delta < \mathrm{cf}(\nu)\rangle$ in ν and define:

$$h_\nu(\alpha) = \begin{cases} \sup_{\delta<\mathrm{cf}(\nu)} h_{\nu_\delta}(\alpha) & \text{if } \mathrm{cf}(\nu) < \kappa \\ \sup_{\delta<\alpha} h_{\nu_\delta}(\alpha) & \text{if } \mathrm{cf}(\nu) = \kappa \end{cases} \tag{1}$$

Then for each $\nu < \kappa^+$, h_ν represents the ν-th canonical function on κ.

We will often use constructions of the following form, where κ is regular and uncountable. Fix some large regular $\theta \geq \kappa^+$ and a wellordering Δ of H_θ. For each $b \in H_\theta$ of cardinality κ let $\mathrm{Filt}^b = \langle \mathrm{Filt}^b_\alpha | \alpha < \kappa\rangle$ denote the Δ-least filtration of b; note this can just be viewed as $\langle f^{b}{}``\alpha | \alpha < \kappa\rangle$ where f^b is some surjection $\kappa \to b$. Let S^b be the collection of $X \in P_\kappa(H_\theta)$ such that $X \prec (H_\theta, \in, \Delta, \{b\})$ and $X \cap \kappa$ is transitive; for each such X let $\alpha_X := X \cap \kappa$. Let $\bar{S}^b := \{\alpha_X | X \in S^b\}$. For each $X \in S^b$ let $\sigma_X : H_X \to H_\theta$ be the inverse of the Mostowski collapsing map for X. Since $b \in X \prec (H_\theta, \Delta, \in)$ then $\mathrm{Filt}^b \in X$ and it is straightforward to show that $X \cap b = \mathrm{Filt}^b_{\alpha_X}$.

Now suppose $|\mathrm{trcl}(b)| < \kappa^+$ (note since we're assuming $|b| = \kappa$, then this means $|\mathrm{trcl}(b)| = \kappa$). If X, Y are both in S^b and $\alpha_X = \alpha_Y =: \alpha$, then $\sigma_X^{-1}(b) = \sigma_Y^{-1}(b)$; this is because $X \cap \mathrm{trcl}(b) = \mathrm{Filt}^{\mathrm{trcl}(b)}_\alpha = Y \cap \mathrm{trcl}(b)$. Thus the following definition does not depend on the particular choice of X:

Definition 2. *If $b \in H_{\kappa^+}$ and $\alpha \in \bar{S}^b$, b_α is defined to be $\sigma_X^{-1}(b)$ where X is* any *element of S^b with $\alpha_X = \alpha$. Sometimes we will refer to the function $\alpha \mapsto b_\alpha$ as the* canonical function indexed by b *and denote it h_b.*

Finally, suppose $\nu \in S^{\kappa^+}_\kappa$, $\langle \nu_\delta | \delta < \kappa\rangle$ is a cofinal sequence in ν, and $X \in S^{\vec{\nu}}$. Then $X \cap \{\nu_\delta | \delta < \kappa\} = \{\nu_\delta | \delta < \alpha_X\}$ and:

(2)
- $\sigma_X^{-1}(\nu) = h_\nu(\alpha_X)$
- $\sigma_X^{-1}(\langle \nu_\delta | \delta < \kappa\rangle) = \langle h_{\nu_\delta}(\alpha_X) | \delta < \alpha_X\rangle$

3. Weakly normal and nonregular ultrafilters

Let U be an ultrafilter on a regular uncountable cardinal κ. Consider the class $\mathcal{G}$ of partial functions $f : \kappa \to V$ such that $\mathrm{dom}(f) \in U$. For functions $f, g \in \mathcal{G}$, $f =_U g$ means that $\{\alpha \in \mathrm{dom}(f) \cap \mathrm{dom}(g) | f(\alpha) = g(\alpha)\} \in U$ (i.e. $\mathrm{ult}(V, U) \models$ "$[f]_U = [g]_U$"). The notations $f \in_U g$ and $f <_U g$ (if f, g are ordinal-valued) are defined similarly.

[4] Unless stated otherwise, whenever we specify a sequence $\langle \nu_\delta | \delta < \mathrm{cf}(\nu)\rangle$, we will always require that it is increasing.

Definition 3. *Let κ be a regular uncountable cardinal, and U a uniform ultrafilter over κ. Let $\lambda < \kappa$ be a cardinal. U is (λ, κ)-nonregular iff whenever $\mathcal{A} \subset U$ and $|\mathcal{A}| = \kappa$ then there is a $\mathcal{A}' \subset \mathcal{A}$ of cardinality λ such that $\bigcap \mathcal{A}' \neq \emptyset$.*

If $\kappa = \lambda^+$, then λ is clearly the largest possible cardinal where U might be (λ, κ)-nonregular. If U is (λ, λ^+)-nonregular then we say it is *fully nonregular*.

Just as nonregularity is a weakened version of κ-completeness of an ultrafilter, the following is a weakened notion of a normal ultrafilter:

Definition 4. *A uniform ultrafilter U over κ is* weakly normal *iff whenever $f : \kappa \to \kappa$ is a regressive (partial) function with dom$(f) \in U$, then there is a $\delta < \kappa$ and $A \in U$ such that $f \restriction A$ is bounded by δ.*

If λ is regular then the existence of a fully nonregular ultrafilter on λ^+ is equivalent to the existence of a weakly normal ultrafilter on λ^+ which concentrates on $S^{\lambda^+}_\lambda$. In fact, every weakly normal ultrafilter concentrating on $S^{\lambda^+}_\lambda$ is fully nonregular, and any fully nonregular ultrafilter projects to a weakly normal ultrafilter in the Rudin-Keisler order, via a "least unbounded function" modulo the nonregular ultrafilter; see Kanamori [10]. Also, if U is uniform and weakly normal, it is easy to see that U extends the club filter.

Lemma 5. *(Diagonalization Lemma). Suppose U is a weakly normal ultrafilter and $X_\xi \in U$ for every $\xi < \kappa$. Then $\{\alpha < \kappa | \alpha \in X_\xi$ for cofinally many $\xi < \alpha\} \in U$.*

Note then if f_ξ is a function with domain $X_\xi \in U$ for each $\xi < \kappa$, then it's natural to define $\text{diagsup}_{\xi<\kappa} f_\xi$ as the function with domain $\{\alpha < \kappa | \alpha \in \textit{domain}(f_\xi)$ for cofinally many $\xi < \alpha\} (\in U)$ which sends $\alpha \mapsto \sup_{\xi<\alpha} f_\xi(\alpha)$. It's easy to see that this diagsup is $\geq_U f_\xi$ for every $\xi < \kappa$.

For the remainder of the section, assume U is a weakly normal ultrafilter on ω_2 and $S^2_1 \in U$.

The proof of Lemma 5.3 in [6] goes through to show:

Lemma 6. *There is* ***no*** *collection $\mathcal{F}$ of functions such that:*

- $|\mathcal{F}| = \omega_3$
- *dom$(f) \in U$ and range$(f) \subset \omega_1$ for every $f \in \mathcal{F}$*
- *$\{\xi \in$ dom$(f) \cap$ dom$(g) | f(\xi) = g(\xi)\}$ is nonstationary for every distinct pair f, g in $\mathcal{F}$.*

The collection $\mathcal{F}$ in the statement of Lemma 6 is often called a "transversal sequence."

Corollary 7. *There is no pair $h, \mathcal{G}$ such that:*

- *$\mathcal{G}$ is an ω_3-sized collection of partial functions from $\omega_2 \to \omega_2$ whose domains are in U;*
- *For every distinct $f, g \in \mathcal{G}$: there are only nonstationarily many $\alpha \in dom(f) \cap dom(g)$ such that $f(\alpha) = g(\alpha)$;*
- *$h : \omega_2 \to \omega_2$ and for every $f \in \mathcal{G}$, $f <_U h$.*

Proof. For each $\beta < \omega_2$ fix an injective $\psi_\beta : \beta \to \omega_1$. For each $f \in \mathcal{G}$, if $\alpha \in dom(f)$ and $f(\alpha) < h(\alpha)$, let $f'(\alpha) := \psi_{h(\alpha)}(f(\alpha))$. Then $\{f' | f \in \mathcal{G}\}$ is an ω_3-sized collection of functions which has the properties listed in Lemma 6, a contradiction. $\square$

The proof of Corollary 5.4 in [6] shows:

Lemma 8. *For every $f : \omega_2 \to \omega_2$, there is a $\nu < \omega_3$ such that $f <_U h_\nu$.*

Finally, the proofs of Lemma 5.5 and Corollary 5.6 in [6] show:

Lemma 9. *If $cf(\nu) = \omega_2$ then $[h_\nu]_U$ is the least upper bound of $\{[h_\tau]_U | \tau < \nu\}$ in $ult(V, U)$.*

4. The Bounding Construction

Throughout the rest of the paper, K denotes the core model for non-overlapping extenders, built under the assumption that 0-pistol does not exist. Basic facts about 0-pistol and K can be found in Chapter 8 of [14]. In particular, K is capable of having a strong cardinal, but comparisons of mice are still *linear*. Note that if 0-pistol exists, then there is a sharp for an inner model with a strong cardinal, so the conclusion of Theorem 1 would hold and we'd be finished.

Let E be K's extender sequence, and for each $X \in P_{\omega_2}(H_{\omega_3})$ such that $X \prec (H_{\omega_3}, \in, E \restriction \omega_3, ...)$ let $K_X := \sigma_X^{-1}[K \cap X]$, where σ_X is the inverse of the Mostowski Collapse of X.

Let S be the collection of $X \in P_{\omega_2}(H_{\omega_3})$ such that $X \prec (H_{\omega_3}, \in, \Delta, ...)$, $\alpha_X := X \cap \omega_2 \in S^2_1$, and $\lambda_X := sup(X \cap \omega_3) \in S^3_1$ (these are sometimes called sets of uniform cofinality ω_1). Then:

(3) $\quad X \cap \omega_3$ is an ω-closed set of ordinals for every $X \in S$.

(so $\sigma_X : H_X \to H_{\omega_3}$ is continuous on cof(ω).) To see that (3) holds, suppose $\nu \in S^3_0 \cap Lim(X)$. Since λ_X has uncountable cofinality, there is some $\nu' \in X \cap \omega_3$ such that $\nu < \nu'$. Then there is a bijection $g : \omega_2 \to \nu'$ such that $g \in X$; since α_X has uncountable cofinality, there is a $\beta < \alpha_X$ such that $(g"\beta) \cap \nu$ is cofinal in ν. But $g"\beta \subset X$, so $X \cap \nu$ is cofinal in ν.

For $\nu \in \omega_3$ consider the set $S \cap S^\nu$. Let $\bar{S}^\nu$ be the projection of $S \cap S^\nu$, i.e. $\bar{S}^\nu := \{\alpha_X | X \in S \cap S^\nu\}$. Note that $\bar{S}^\nu$ contains an unbounded subset which is closed under limits of cofinality ω_1.[5] Since U extends the club filter and $S^2_1 \in U$ (by assumption), then $\bar{S}^\nu \in U$. For $X \in S^\nu$:

(4) σ^ν_X denotes the map $\sigma_X \restriction h_\nu(\alpha_X)$.

Note that for any X which has the sequence $\langle \nu_\delta | \delta < cf(\nu) \rangle$ as an element,[6] it holds that $h_\nu(\alpha_X) = otp(X \cap \nu) = \sigma_X^{-1}(\nu)$.

If X, Y are both in S^ν and $\alpha_X = \alpha_Y =: \alpha$, then $X \cap \nu = Y \cap \nu$ (they both equal Filt^ν_α). So $\sigma^\nu_X = \sigma^\nu_Y$. In other words, the map σ^ν_X is independent of our choice of $X \in S^\nu$ with $\alpha_X = \alpha$. So the following definition makes sense:

Definition 10. *If $\nu \in \omega_3$ and $\alpha \in \bar{S}^\nu$, we define σ^ν_α as the map σ^ν_X, where X is* any *element of S^ν with $\alpha = \alpha_X$.*

Now for any $X \in S$, consider the coiteration of K_X with K. Let Ω_X denote the length of the K_X vs. K coiteration, and

$$\langle N^X_i, \pi^X_{i,j}, \mu^X_i, \kappa^X_i, \tau^X_i, \delta^X_i, (N^X_i)^* | i \leq j \leq \Omega_X \rangle$$

denote the objects on the K side of the coiteration; i.e. N^X_i is the i-th iterate, $\pi^X_{i,j}$ the iteration map, μ^X_i the iteration index, κ^X_i the critical point, τ^X_i the smaller of the cardinal successors of κ^X_i in the i-th iterates, δ^X_i the maximal segment of N^X_i where τ^X_i is a cardinal, and $(N^X_i)^* = N^X_i || \delta^X_i$.

Since each $X \in S$ has an ω-closed intersection with ω_3 (see (3)), Lemma 39 from [2] applies,[7] so $\alpha_X^{+K_X}$ is not a cardinal in K, the K side of the coiteration truncates to an ω_1-sized mouse either at stage 0 or at stage 1, and the K_X side of the coiteration is trivial. Let $\iota^X_0 \in \{0,1\}$ denote the first truncation stage. So $|(N^X_{\iota^X_0})^*| < \omega_2$. Since $|(N^X_{\iota^X_0})^*| < \omega_2$, and $|K_X| < \omega_2$, then $|(N^X_i)^*| < \omega_2$ for every $i \in [\iota^X_0, \Omega_X)$, and $\Omega_X < \omega_2$. Also, since the K_X side is trivial in the coiteration, we have:

$$\mu^X_i = o^{K_X}(\kappa^X_i) \tag{5}$$

for every stage i of the coiteration.

For $\nu \in \omega_3$ and $X \in S^\nu$ let $K^\nu_X = K_X | \sigma_X^{-1}(\nu)$. Let θ^ν_X denote the least ordinal ι such that μ^X_ι is either not defined or is at least

[5]This is standard; construct an elementary $\in$-chain $\langle X_\beta | \beta < \omega_2 \rangle$ consisting of models from $S \cap S^\nu$, such that for β of cofinality ω_1, $X_\beta = \bigcup_{\beta' < \beta} X_{\beta'}$. Then the projection of this chain contains an ω_1-club in ω_2.

[6]where this $\vec{\nu}$ was the cofinal sequence in ν used to define the "official" representative h_ν for the ν-th canonical function in (1).

[7]This was an argument due to Mitchell.

height$(K^\nu_X) = h_\nu(\alpha_X)$. In other words: θ^ν_X is equal to the length of the K vs. K^ν_X coiteration, unless the height of K^ν_X (which is $h_\nu(\alpha_X)$) indexes an extender in $N^X_{\theta^\nu_X}$; in this latter case the length of the K vs. K^ν_X coiteration will equal $\theta^\nu_X + 1$. Define $N^\nu_X := N^\nu_{\theta^\nu_X}$. Similarly to the discussion before Definition 10, if $\alpha \in \bar{S}^\nu$ then the object K^ν_X does not depend on the particular choice of $X \in S^\nu$ with $\alpha_X = \alpha$. So the following definition makes sense:

Definition 11. *If $\nu \in \omega_3$ and $\alpha \in \bar{S}^\nu$: define $K^\nu_\alpha, N^\nu_\alpha, \theta^\nu_\alpha$ as $K^\nu_X, N^\nu_X, \theta^\nu_X$ (respectively) where X is* any *element of S^ν with $\alpha = \alpha_X$.*

Given any $\nu < \nu'$ in ω_3, K^ν_α is an initial segment of $K^{\nu'}_\alpha$ for all but nonstationarily many $\alpha \in S^2_1$, and neither move in their coiteration with K. Then the coiteration of K with K^ν_α is an initial segment of the coiteration of K with $K^{\nu'}_\alpha$. This motivates the next definition:

Definition 12. *Let $\nu < \nu'$ and $\alpha \in \bar{S}^\nu \cap \bar{S}^{\nu'}$. Let M be any mouse such that $K^{\nu'}_\alpha$ is an initial segment of M, and let $\pi_{i,j}$ denote the iteration maps on the K side of the K vs. M coiteration. $\pi^{\nu,\nu'}_\alpha$ will denote the (possibly partial) iteration map $\pi_{\theta^\nu_\alpha, \theta^{\nu'}_\alpha}$.*

Of course, a priori it could happen that $\theta^\nu_\alpha = \theta^{\nu'}_\alpha$; we will see this is not usually the case.

Suppose $\nu \in S^3_2$ and $t_\nu = \langle \nu_\delta | \delta < \omega_2 \rangle$ is a (not necessarily continuous) sequence cofinal in ν. Pick any $X \in S^{t_\nu}$. Then $X \cap \{\nu_\delta | \delta < \omega_2\} = \{\nu_\delta | \delta < \alpha_X\}$; so $X \in S^{\nu_\delta}$ for every $\delta < \alpha_X$, and $X \cap \nu_\delta = \mathrm{Filt}^{\nu_\delta}_{\alpha_X}$. Then

- For every $\delta < \alpha_X$: $K^{\nu_\delta}_{\alpha_X} = \sigma^{-1}_X(K|\nu_\delta)$ is an initial segment of $K^\nu_{\alpha_X} = \sigma^{-1}_X(K|\nu)$ and the height of $K^{\nu_\delta}_{\alpha_X}$ is $\mathrm{otp}(\mathrm{Filt}^{\nu_\delta}_{\alpha_X}) = h_{\nu_\delta}(\alpha_X)$;
- The height of $K^\nu_{\alpha_X}$ is $\mathrm{otp}(\mathrm{Filt}^\nu_{\alpha_X}) = h_\nu(\alpha_X)$ and $K^\nu_{\alpha_X} = \bigcup_{\delta<\alpha_X} K^{\nu_\delta}_{\alpha_X}$

Since the K_X side of the coiteration is trivial and $K^\nu_X = \bigcup_{\delta<\alpha_X} K^{\nu_\delta}_X$, then $\theta^\nu_X = \sup_{\delta<\alpha_X} \theta^{\nu_\delta}_X$. This follows from the way we defined the ordinal θ^ν_X; it's possible that the length of the coiteration of K with K^ν_X is actually $\theta^\nu_X + 1$, which may happen if $h_\nu(\alpha_X) = ht(K^\nu_X)$ is a coiteration index. However, we will see later that this does not happen if K has no overlapping extenders.

Let $\nu < \omega_3$. Recall for every $\alpha \in \bar{S}^\nu$, $\alpha^{+K^\nu_\alpha}$ is not a cardinal in K and there is a truncation by stage 1 on the K side. In particular, $(N^\nu_\alpha)^*$ has cardinality $< \omega_2$. Let $\mathcal{Q}^\nu_\alpha$ denote the collection of all mice which are iterates of $(N^\nu_\alpha)^*$ where the iteration is of length ≤ 2; i.e. the collection of all possible iterates of $(N^\nu_\alpha)^*$ for at most 2 stages past stage θ^ν_α . Let $q^\nu_\alpha := \sup\{\mathrm{height}(N) | N \in \mathcal{Q}^\nu_\alpha\}$; note $q^\nu_\alpha < \omega_2$. Define

(6) $$\psi_\nu : \bar{S}^\nu \to \omega_2 \text{ by } \alpha \mapsto q^\nu_\alpha$$

The definition of ψ_ν implies that whenever ν' is larger than ν, $\psi_\nu(\alpha) < h_{\nu'}(\alpha)$, and $\alpha = \alpha_X$ for some $X \in S^\nu \cap S^{\nu'}$, then the coiteration of K with $K^{\nu'}_\alpha$ is properly longer than the coiteration of K with K^ν_α; in fact, for such ν, ν', and α we have:

(7)
- The length of the K vs. $K^{\nu'}_\alpha$ coiteration is at least $\theta^\nu_\alpha + 2$
- $\kappa^{\nu'}_{\theta^\nu_\alpha+1} > \mu^{\nu'}_{\theta^\nu_\alpha} \geq \text{height}(K^\nu_\alpha) = h_\nu(\alpha)$.

Now use Lemma 8 to arrange exactly that situation; i.e. run the "bounding construction" similar to that in [4]. There is a key difference between the construction here and that in [4]: in the current construction, we have not yet made any assumptions about the K-strength of the ordinals in $S^3_2 \cap K$, as this is not needed for the bounding construction. Recursively define a closed unbounded subset D of ω_3 as follows: suppose some initial segment $\bar{D}$ of D has been defined. If $\bar{D}$ has no largest element, we choose $\sup(\bar{D})$ to be the next element of D. Otherwise $\bar{D}$ has a largest element, say ν. Then the next element of D—which we will denote ν^*—is chosen so that $\psi_\nu <_U h_{\nu^*}$; this is possible by Lemma 8.

Then for every $\nu \in D$ there are U-many α such that $\psi_\nu(\alpha) < h_{\nu^*}(\alpha)$. This is the scenario described before (7), so

$$\theta^\nu_{(-)} <_U \theta^{\nu^*}_{(-)} \text{ (in fact } \theta^\nu_{(-)} + 1 <_U \theta^{\nu^*}_{(-)}) \tag{8}$$

Here, the symbol $\theta^\nu_{(-)}$ denotes the (partial) function $\alpha \mapsto \theta^\nu_\alpha$. We will also use the notation $\{\theta^\nu_{(-)} < \theta^{\nu^*}_{(-)}\}$ to denote the set of α such that both θ^ν_α, $\theta^{\nu^*}_\alpha$ are defined and $\theta^\nu_\alpha < \theta^{\nu^*}_\alpha$ (and similarly for other functions on ω_2).

Then for $\alpha \in \{\theta^\nu_{(-)} < \theta^{\nu^*}_{(-)}\}$ let $\kappa^\nu_\alpha := \kappa^X_{\theta^\nu_\alpha}$ where $X \in S^\nu \cap S^{\nu^*}$ and $\alpha = \alpha_X$—i.e. the critical point at stage θ^ν_α of the K versus $K^{\nu^*}_\alpha$ coiteration. Note that if ν' is some element of D which is at least ν^* and $\alpha = \alpha_X$ for some $X \in S^{\nu'} \cap S^\nu \cap S^{\nu^*}$ then κ^ν_α is also the critical point at stage θ^ν_α of the K vs. $K^{\nu'}_X$ coiteration. Similarly define μ^ν_α as the iteration index at stage θ^ν_α of the K vs. $K^{\nu^*}_\alpha$ coiteration. Note that there may be a truncation at stage θ^ν_α. We will arrange that, often enough, such truncations do not occur; this is not essential to the argument but it will provide some simplifications.

The construction of D guarantees that for every $\nu \in D$:

$$h_\nu \leq_U \psi_\nu <_U h_{\nu^*} \leq_U \psi_{\nu^*} \tag{9}$$

and

$$h_\nu \leq_U \psi_\nu <_U \kappa^{\nu^*}_{\theta^\nu_{(-)}+1} \leq_U \kappa^{\nu^*}_{\theta^{\nu^*}_{(-)}} =_{\text{by def.}} \kappa^{\nu^*}_{(-)} <_U \psi_{\nu^*} \tag{10}$$

Lemma 9 then guarantees that for every $\nu \in \mathrm{Lim}(D) \cap S^3_2$, h_ν is the least upper bound of $\{\kappa^\tau_{(-)} | \tau \in D \cap \nu\}$ in the ordering $<_U$. Since clearly $\kappa^\nu_{(-)} \geq_U \kappa^\tau_{(-)}$ for every $\tau \in D \cap \nu$, then

$$h_\nu \leq_U \kappa^\nu_{(-)} \text{ for every } \nu \in \mathrm{Lim}(D) \cap S^3_2 \tag{11}$$

We will eventually show that whenever R is a stationary subset of $D \cap S^3_2$, there are many $\nu \in R$ such that $h_\nu =_U \kappa^\nu_{(-)}$.

Also, we note that whenever $\nu < \nu' < \nu''$ are all in D, then

$$h_\nu <_U \kappa^{\nu'}_{(-)} <_U h_{\nu''}. \tag{12}$$

To see (12): since $\nu^* \leq \nu'$ and $(\nu')^* \leq \nu''$ the bounding construction yields $h_\nu <_U \kappa^{\nu^*}_{(-)} \leq_U \kappa^{\nu'}_{(-)} <_U h_{(\nu')^*} \leq_U h_{\nu''}$.

Lemma 13. *For every $\nu \in Lim(D) \cap S^3_2$:*

1. *There are U-many α such that θ^ν_α is a limit ordinal.*
2. *Let $t_\nu = \langle \nu_\delta | \delta < \omega_2 \rangle$ be a sequence of members of D which is cofinal in ν. Then there are U-many α such that*
   - *(a) $h_\nu(\alpha) = sup_{\delta<\alpha} \kappa^{\nu_\delta}_\alpha$*
   - *(b) $height(N^{\nu_\delta}_\alpha) < h_\nu(\alpha)$ for all sufficiently large $\delta < \alpha$.*

Proof. First prove part 1. Fix any $t_\nu = \langle \nu_\delta | \delta < \omega_2 \rangle$ which is a sequence of members of D which is cofinal in ν. Recall that the K^ν_α side of the coiteration is trivial and $\theta^\nu_\alpha = \sup_{\delta<\alpha} \theta^{\nu_\delta}_\alpha$ is the least stage of the K vs. K^ν_α coiteration where the iteration index is at least $\mathrm{ht}(K^\nu_\alpha)$ (or the length of the K vs. K^ν_α coiteration, if there is no such index). For each $\delta < \omega_2$, let $B_\delta := \{\alpha | \theta^{\nu_\delta}_\alpha < \theta^{\nu_{\delta+1}}_\alpha\}$; this is an element of U by (8) (note the least element of D above ν_δ is $\leq \nu_{\delta+1}$). Then $B := \{\alpha | \alpha \in B_\delta \text{ for cofinally many } \delta < \alpha\}$ is an element of U by Lemma 5. Then if $\alpha \in B$, the sequence $\langle \theta^{\nu_\delta}_\alpha | \delta < \alpha \rangle$ does not stabilize; so $\theta^\nu_\alpha = \sup_{\delta<\alpha} \theta^{\nu_\delta}_\alpha$ is a limit ordinal.

The proof of the part 2 is similar. Fix any $\langle \nu_\delta | \delta < \omega_2 \rangle$ which is contained in D and cofinal in ν. By (12), the set $B_\delta := \{\alpha | h_{\nu_\delta}(\alpha) < \kappa^{\nu_{\delta+1}}_\alpha < h_{\nu_{\delta+2}}(\alpha)\}$ is an element of U. By Lemma 5, $B := \{\alpha | \alpha \in B_\delta \text{ for cofinally many } \delta < \alpha\}$ is an element of U. Pick any $\alpha \in B$; then $h_{\nu_\delta}(\alpha) < \kappa^{\nu_{\delta+1}}_\alpha < h_{\nu_{\delta+2}}(\alpha)$ for cofinally many $\delta < \alpha$, so $\sup_{\delta<\alpha} h_{\nu_\delta}(\alpha) \leq \sup_{\delta<\alpha} \kappa^{\nu_{\delta+1}}_\alpha \leq \sup_{\delta<\alpha} h_{\nu_{\delta+2}}(\alpha)$. But the left and right sides of this last inequality are both $h_\nu(\alpha)$, so $h_\nu(\alpha) = \sup_{\delta<\alpha} \kappa^{\nu_\delta}_\alpha$.

Finally, we find U-many α such that $\mathrm{height}(N^{\nu_\delta}_\alpha) < h_\nu(\alpha)$ for sufficiently large $\delta < \alpha$. Let $\delta < \omega_2$. The definition of ψ_{ν_δ} and the bounding construction imply $\mathrm{height}(N^{\nu_\delta}_{(-)}) \leq_U \psi_{\nu_\delta} <_U h_{\nu_{\delta+1}}$; and since $h_{\nu_\delta} <_U h_\nu$ (in fact $h_{\nu_\delta}(\alpha) < h_\nu(\alpha)$ for all but nonstationarily many $\alpha \in S^2_1$), in particular $E_\delta := \{\alpha | \mathrm{height}(N^{\nu_\delta}_\alpha) < h_\nu(\alpha)\}$ is an element of U. By

Lemma 5 there are U-many α such that $\alpha \in E_\delta$ for cofinally many $\delta < \alpha$. Then for such α we have height$(N^{\nu_\delta}_\alpha) < h_\nu(\alpha)$ for cofinally many $\delta < \alpha$. Since $\theta^\nu_\alpha = \sup_{\delta<\alpha}\theta^{\nu_\delta}_\alpha$ is a limit ordinal (by part 1), then in fact height$(N^{\nu_\delta}_\alpha) < h_\nu(\alpha)$ for all sufficiently large $\delta < \alpha$. $\square$

So (11) and Lemma 13 imply that for all but nonstationarily many $\nu \in S^3_2$, if $\langle \nu_\delta | \delta < \omega_2\rangle$ is cofinal in ν then there are U-many α such that:

$$\sup_{\delta<\alpha}\kappa^{\nu_\delta}_\alpha = h_\nu(\alpha) \leq \kappa^\nu_\alpha \tag{13}$$

We will eventually replace the inequality in (13) with an equality (mod U).

Much of the theory in the remainder of this section is implicit in [6]. A subtle but important difference, however, is that in the current paper we arrange that certain properties hold on V-*clubs* of ordinals in S^3_2, rather than just on stationary sets; this is essential to the proof in the later sections.

Definition 14. *A* system of objects from the winning mouse *is a sequence of functions $\mathcal{S} = \langle Obj^\nu_{(-)} | \nu \in R\rangle$ where $R \subset S^3_2$ is stationary and for every $\nu \in R$: $dom(Obj^\nu_{(-)}) \in U$ and $Obj^\nu_{(-)} \in_U N^\nu_{(-)}$.*

Lemma 15. *Suppose $R \subseteq Lim(D) \cap S^3_2$ and $\mathcal{S} = \langle Obj^\nu_{(-)} | \nu \in R\rangle$ is a system of objects from the winning mouse. Then there is a stationary $R' \subset R$, an ordinal $\bar\tau < \omega_3$ such that: for every $\nu \in R'$ there are U-many α where*

(14)
(1) There is no truncation at any coiteration stages in the interval $[\theta^{\bar\tau}_\alpha, \theta^\nu_\alpha)$
(2) Obj^ν_α is in the range of the iteration map $\pi^{\bar\tau,\nu}_\alpha$ (see Definition 12 for the meaning of $\pi^{\bar\tau,\nu}_\alpha$).
(3) The preimage of Obj^ν_α by $\pi^{\bar\tau,\nu}_\alpha$ appears in the corresponding mouse strictly below level $h_{\bar\tau}(\alpha)$ of that mouse.

Proof. For each $\nu \in R$ fix some $t_\nu = \langle \nu_\delta | \delta < \omega_2\rangle$ which is a sequence of points in D which is cofinal in ν. By Lemma 13, $\theta^\nu_\alpha = \sup_{\delta<\alpha}\theta^{\nu_\delta}_\alpha$ is a limit ordinal for every $\nu \in D \cap S^3_2$ (for U-many α); so there is some $\delta^\nu(\alpha) < \alpha$ such that a thread to Obj^ν_α appears by stage $\theta^{\nu_{\delta^\nu(\alpha)}}_\alpha$ and there are no truncations for stages in the interval $[\theta^{\nu_{\delta^\nu(\alpha)}}_\alpha, \theta^\nu_\alpha)$. More precisely: letting $\bar\theta := \theta^{\nu_{\delta^\nu(\alpha)}}_\alpha$ then Obj^ν_α is in the range of the iteration map $\pi^X_{\bar\theta,\theta^\nu_\alpha}$ whenever $X \in S^\nu$ and $\alpha = \alpha_X$. By the weak normality of U, for each $\nu \in R$ there is a $\bar\delta^\nu < \omega_2$ and a set $A_{t_\nu} \in U$ such that $\delta^\nu_\alpha \leq \bar\delta^\nu$ for every $\alpha \in A_{t_\nu}$; so the thread to Obj^ν_α appears by stage $\theta^{\nu_{\bar\delta^\nu}}_\alpha$ for every

$\alpha \in A_{t_\nu}$. The point is that for $\alpha \in A_{t_\nu}$, the superscript on the stage $\theta_\alpha^{\nu_{\bar\delta^\nu}}$ no longer depends on α.

Since $R \subset \omega_3$ is stationary and $\bar\delta^\nu \in \omega_2$, there is a stationary $\tilde R \subset R$ and a fixed $\bar\delta$ such that $\bar\delta^\nu = \bar\delta$ for every $\nu \in \tilde R$. In other words, for every $\nu \in \tilde R$ and every $\alpha \in A_{t_\nu}$, a thread to Obj_α^ν appears by stage $\theta_\alpha^{\nu_{\bar\delta}}$.

So for each $\nu \in \tilde R$ define the following functions with domain A_{t_ν}:

$$\begin{aligned} \bar\theta_\alpha^\nu &:= \theta_\alpha^{\nu_{\bar\delta}} \\ \phi^\nu(\alpha) &:= \text{ the preimage of } \mathrm{Obj}_\alpha^\nu \text{ by } \pi^\nu_{\bar\theta_\alpha^\nu, \theta_\alpha^\nu} \end{aligned} \tag{15}$$

Lemma 13—specifically the part about height$(N_\alpha^{\nu_\delta})$—implies for every $\nu \in \tilde R$:

$$\phi^\nu <_U h_\nu \tag{16}$$

Lemma 9 and (16) imply that for every $\nu \in \tilde R$ there is a $\tau(\nu) < \nu$ such that $\phi^\nu <_U h_{\tau(\nu)}$. So an application of the Fodor Lemma yields a fixed $h_{\bar\tau}$ and a stationary $R' \subset \tilde R$ such that $\phi^\nu <_U h_{\bar\tau}$ for every $\nu \in R'$. □

Corollary 16. *For all but nonstationarily many $\nu \in D \cap S_2^3$: There are U-many α such that there are no truncations at any stage in the interval $[\theta_\alpha^\nu, \theta_\alpha^{\nu^*}]$.*

Proof. Suppose to the contrary that there is a stationary $R \subseteq \mathrm{Lim}(D) \cap S_2^3$, such that for each $\nu \in R$ there are U-many α with some truncation in the interval $[\theta_\alpha^\nu, \theta_\alpha^{\nu^*}]$; let $A_\nu \in U$ be the collection of such α. Let $R' \subset R$ and $\bar\tau < \omega_3$ be as in the conclusion of Lemma 15. Pick any pair $\bar\nu < \nu$ which are both in R' and such that $\bar\tau < \bar\nu < \bar\nu^* < \nu$ (recall the star superscript indicates the next element of D above the point). Let $B_\nu \in U$ be a collection of α where (14) holds. Now $\bar\tau < \bar\nu < \bar\nu^* < \nu$ and these are all in D, so by (8) the set $C := \{\alpha | \theta_\alpha^{\bar\tau} < \theta_\alpha^{\bar\nu} < \theta_\alpha^{\bar\nu^*} < \theta_\alpha^\nu\}$ is an element of U. So $A_\nu \cap B_\nu \cap C \in U$; pick any α in this intersection. Since $\alpha \in B_\nu$ then there are no truncations in $[\theta_\alpha^{\bar\tau}, \theta_\alpha^\nu)$. But this contradicts that $\alpha \in A_\nu \cap C$. □

Definition 17. *Let $\mathcal{S} = \langle Obj_{(-)}^\nu | \nu \in R \rangle$ be a system of objects from the winning mouse. Let $R' \subset R$. We say that $\mathcal{S}$* lines up *on R' iff for every $\bar\nu < \nu$ in R':*

- *There are U-many $\alpha \in \bar S^{\bar\nu} \cap \bar S^\nu$ such that there are no truncations at any coiteration stages in the interval $[\theta_\alpha^{\bar\nu}, \theta_\alpha^\nu]$*
- *$\theta_{(-)}^{\bar\nu} <_U \theta_{(-)}^\nu$*
- *$Obj_{(-)}^\nu \in_U range(\pi_{(-)}^{\bar\nu,\nu})$*
- *$\pi_{(-)}^{\bar\nu,\nu}(Obj_{(-)}^{\bar\nu}) =_U Obj_{(-)}^\nu$*

If we specify that $\mathrm{Obj}^\nu_\alpha := \kappa^\nu_\alpha$—i.e. if the system of objects from the winning mouse are chosen to be the first critical point beyond the K vs. K^ν_α coiteration—for every ν and α, then as in [6] these objects line up nicely (Lemma 18 below). Later, in Section 6 under the additional assumption that $|{}^{\omega_2}\omega_1/U|$ is small, we will choose the objects of interest to be extenders, and use the small ultrapower assumption to get the objects to line up nicely.

Lemma 18. *Same assumptions as Lemma 15. Let $R' \subset R$ be the stationary set from the conclusion of that Lemma.*

If the system of objects from the winning mouse are just the critical points (i.e. $Obj^\nu_{(-)} =_U \kappa^\nu_{(-)}$ for each $\nu \in R$) then there is a stationary $R'' \subseteq R'$ such that the system lines up on R''.

Proof. First define the function $\bar{\phi}^\nu$ as follows; it is similar to the function ϕ^ν from the proof of Lemma 15, but this new definition takes advantage of the fixed ordinal $\bar{\tau}$ obtained in that lemma:

$$\bar{\phi}^\nu(\alpha) \;:=\; \text{the preimage of } \kappa^\nu_\alpha \text{ by } \pi^\nu_{\theta^{\bar{\tau}}_\alpha, \theta^\nu_\alpha} \tag{17}$$

Then for every $\nu < \nu'$ in R': $\bar{\phi}^\nu \leq_U \bar{\phi}^{\nu'}$; in fact:

(18) The collection of α such that $\bar{\phi}^\nu(\alpha) > \bar{\phi}^{\nu'}(\alpha)$ is nonstationary

since whenever $X \in S^\nu \cap S^{\nu'}$ then the coiteration of K with $K^\nu_{\alpha_X}$ is an "initial segment" of the coiteration of K with $K^{\nu'}_{\alpha_X}$, so $\bar{\phi}^\nu(\alpha_X) \leq \bar{\phi}^{\nu'}(\alpha_X)$. This uses the fact that the objects of interest are the critical points of the coiteration; it is where the proof breaks down if the objects of interest are, e.g., extenders (though in Section 6 we will do exactly that, with an additional assumption on U).

Then:

$$\langle \bar{\phi}^\nu | \nu \in R' \rangle \text{ is eventually constant modulo } U \tag{19}$$

To see (19): suppose this failed. If $\nu < \nu'$ are in R' and $\bar{\phi}^\nu \neq_U \bar{\phi}^{\nu'}$, then by (18) and the fact that U extends the club filter, the only possibility is that $\bar{\phi}^\nu <_U \bar{\phi}^{\nu'}$. So the failure of (19) implies there is an unbounded $\tilde{R}' \subset R'$ (not necessarily stationary) such that $\langle \bar{\phi}^\nu | \nu \in \tilde{R}' \rangle$ is a $<_U$-increasing sequence.

For each $\nu \in \tilde{R}'$ let $\tilde{\nu}$ denote the least element of $\tilde{R}'$ above ν. Let $\tilde{\phi}^\nu := \bar{\phi}^\nu \restriction \{\alpha \in \mathrm{dom}(\bar{\phi}^\nu) \cap \mathrm{dom}(\bar{\phi}^{\tilde{\nu}}) | \bar{\phi}^\nu(\alpha) < \bar{\phi}^{\tilde{\nu}}(\alpha)\}$ (note then $\mathrm{dom}(\tilde{\phi}^\nu) \in U$). Then $\tilde{\phi}^\nu <_U \tilde{\phi}^{\tilde{\nu}}$ and (18) implies that whenever $\nu < \nu'$ are in $\tilde{R}'$ then $\tilde{\phi}^\nu$ and $\tilde{\phi}^{\nu'}$ agree at only nonstationarily many points.

Furthermore, there is a fixed function from $\omega_2 \to \omega_2$—namely, $h_{\bar{\tau}}$—which U-bounds $\tilde{\phi}^\nu$ for every $\nu \in \tilde{R}'$. This contradicts Corollary 7, and completes the proof of (19).

Let R'' be the tail end of R' on which $\bar{\phi}_\nu$ are pairwise equal modulo U. Then R'' is the desired set. □

Lemma 19. *Assume $R \subset S^3_2$ is stationary. Let R'' be a stationary subset of R on which $\langle \kappa^\nu_{(-)} | \nu \in R \rangle$ lines up (such an R'' exists from the conclusion of Lemma 18). Then $\kappa^\nu_{(-)} =_U h_\nu$ for every $\nu \in R'' \cap Lim(R'')$.*

Proof. Fix any sequence $t_\nu = \langle \nu_\delta | \delta < \omega_2 \rangle$ of points in R'' which is cofinal in ν. Let $A_{t_\nu} \in U$ be a set which witnesses the conclusion of Lemma 13. For each $\delta < \omega_2$, since ν_δ, ν are both in R'' and the system lines up on R'', there is a set $A_{\nu_\delta,\nu} \in U$ which witnesses this fact (so $\pi^{\nu_\delta,\nu}_\alpha(\kappa^{\nu_\delta}_\alpha) = \kappa^\nu_\alpha$ for every $\alpha \in A_{\nu_\delta,\nu}$). By Lemma 5 the set $\{\alpha | \alpha \in A_{\nu_\delta,\nu} \cap A_{t_\nu}$ for cofinally many $\delta < \alpha\}$ is an element of U; for such α, let $I_\alpha := \{\delta < \alpha | \pi^{\nu_\delta,\nu}_\alpha(\kappa^{\nu_\delta}_\alpha) = \kappa^\nu_\alpha\}$. Then $\kappa^\nu_\alpha = \pi^{\nu_{\bar\delta},\nu}_\alpha(\kappa^{\nu_{\bar\delta}}_\alpha) = \sup_{\delta<\alpha} \kappa^{\nu_\delta}_\alpha$ for each $\bar\delta \in I_\alpha$ (since I_α is cofinal in α). Since $\alpha \in A_{t_\nu}$ then $\sup_{\delta<\alpha} \kappa^{\nu_\delta}_\alpha = h_\nu(\alpha)$. So $\kappa^\nu_\alpha = h_\nu(\alpha)$. □

In particular, since R was assumed to be any stationary subset of S^3_2 then:

Corollary 20. *$\kappa^\nu_{(-)} =_U h_\nu$ for all but nonstationarily many $\nu \in S^3_2$.*

Corollary 21. *Suppose for each $\nu < \omega_3$, b^ν is a subset of ν which is an element of K. Recall from Definition 2 that b^ν_α is defined as $\sigma^{-1}_X(b^\nu)$ where X is any element of S^{b^ν} such that $\alpha = \alpha_X$ and $b^\nu \in X$. Then $b^\nu_{(-)} \in_U N^\nu_{(-)}$ for all but nonstationarily many $\nu \in S^3_2$.*

Proof. Since $b^\nu \in \wp^K(\nu)$ then $b^\nu_{(-)} \in_U \wp^{K^\nu_{(-)}}(h_\nu(-)) =_U \wp^{K^\nu_{(-)}}(\kappa^\nu_{(-)})$; the last equality is by Corollary 20. Since the K^ν_α side of the coiteration with K is simple (in fact trivial), then $\wp^{K^\nu_{(-)}}(\kappa^\nu_{(-)}) \subseteq_U \wp^{N^\nu_{(-)}}(\kappa^\nu_{(-)})$. So $b^\nu_{(-)} \in_U \wp^{N^\nu_{(-)}}(\kappa^\nu_{(-)})$. □

Corollary 22. *Assume $b \in \wp^K(\omega_3)$ and R is a stationary subset of S^3_2. Let $b^\nu := b \cap \nu$ for each $\nu < \omega_3$. By Corollary 21, letting $R_b := \{\nu \in S^3_2 | b^\nu_{(-)} \in_U N^\nu_{(-)}\}$, $R - R_b$ is nonstationary, so $\mathcal{S}_b := \langle (\kappa^\nu_{(-)}, b^\nu_{(-)}) | \nu \in R_b \rangle$ is a system of objects from the winning mouse.*

Then there is a stationary $R''_b \subset R_b$ on which $\mathcal{S}_b$ lines up.

Proof. Let R'_b be a stationary subset of R_b whose existence is guaranteed by Lemma 15, and let $\bar\tau$ be the fixed ordinal from the conclusion of that lemma; so $b^\nu_{(-)} \in_U \text{range}(\pi^{\bar\tau,\nu}_{(-)})$ for every $\nu \in R'_b$.

Using Lemma 18, refine R'_b further to a stationary R''_b on which the critical points align; i.e. such that $\langle \kappa^\nu_{(-)} | \nu \in R'_b \rangle$ lines up on R''_b.

Pick any $\bar{\nu} < \nu$ both in R''_b; WLOG assume $\bar{\tau} < min(R''_b)$. Since $R''_b \subset R'_b$ and $\bar{\tau} < \bar{\nu} < \nu$, then

$$b^\nu_\alpha \in \text{range}(\pi^{\bar{\nu},\nu}_\alpha) \text{ for } U\text{-many } \alpha. \tag{20}$$

Let $A_{\bar{\nu},\nu} \in U$ be the collection of such α.

Now consider any $X \in S^{\bar{\nu}} \cap S^\nu \cap S^b$, and let $\sigma_X : H_X \to H_\theta$ be the inverse of the Mostowski collapsing map for X. Let $\alpha = X \cap \omega_2$. Then

$$\begin{aligned} & b^{\bar{\nu}}_\alpha = b^{\bar{\nu}}_X = \sigma_X^{-1}(b^{\bar{\nu}}) = \sigma_X^{-1}(b \cap \bar{\nu}) = \sigma_X^{-1}(b \cap \nu \cap \bar{\nu}) = \\ & \sigma_X^{-1}(b^\nu \cap \bar{\nu}) = \sigma_X^{-1}(b^\nu) \cap \sigma_X^{-1}(\bar{\nu}) = b^\nu_\alpha \cap h_{\bar{\nu}}(\alpha) \end{aligned} \tag{21}$$

Let $B_{\bar{\nu},\nu}$ be the collection of such α; note $B_{\bar{\nu},\nu}$ is almost all of S^2_1 and is thus an element of U.

Let $C_{\bar{\nu},\nu} \in U$ be the collection of α such that

$$\kappa^{\bar{\nu}}_\alpha = h_{\bar{\nu}}(\alpha) \text{ and } \kappa^\nu_\alpha = h_\nu(\alpha) \tag{22}$$

(Note that by Corollary 20 we can WLOG assume that $\kappa^\tau_{(-)} =_U h_\tau$ for every $\tau \in R_b$).

Finally, let $D_{\bar{\nu},\nu}$ be the collection of α such that $\pi^{\bar{\nu},\nu}(\kappa^{\bar{\nu}}_\alpha) = \kappa^\nu_\alpha$; this set is in U because $\langle \kappa^\nu_{(-)} | \nu \in R'_b \rangle$ lines up on R''_b.

Then $\pi^{\bar{\nu},\nu}_\alpha(b^{\bar{\nu}}_\alpha) = b^\nu_\alpha$ for any $\alpha \in A_{\bar{\nu},\nu} \cap B_{\bar{\nu},\nu} \cap C_{\bar{\nu},\nu} \cap D_{\bar{\nu},\nu}$. □

Corollary 23. *Same assumptions as Corollary 22; let R''_b be as in the conclusion of that corollary. Then for every $\nu \in R''_b \cap Lim(R''_b)$ and every sequence $s_\nu = \langle \nu_\delta | \delta < \omega_2 \rangle$ of points in R''_b which is cofinal in ν, there is a set $G^b_{s_\nu} \in U$ such that for every $\alpha \in G^b_{s_\nu}$:*

$$h_\nu(\alpha) = \kappa^\nu_\alpha \text{ and for cofinally many } \delta < \alpha\text{:} \quad \begin{array}{l} \bullet\ h_{\nu_\delta}(\alpha) = \kappa^{\nu_\delta}_\alpha \\ \bullet\ \pi^{\nu_\delta,\nu}_\alpha(\kappa^{\nu_\delta}_\alpha) = \kappa^\nu_\alpha \\ \bullet\ \pi^{\nu_\delta,\nu}_\alpha(b^{\nu_\delta}_\alpha) = b^\nu_\alpha \end{array} \tag{23}$$

Proof. Fix a $\nu \in R''_b \cap \mathrm{Lim}(R''_b)$ and cofinal sequence $s_\nu = \langle \nu_\delta | \delta < \omega_2 \rangle$ ($\subset R''_b$) as in the statement of the corollary. For each $\delta < \omega_2$ let $G^b_{\nu_\delta,\nu}$ be the collection of α such that:

- $h_{\nu_\delta}(\alpha) = \kappa^{\nu_\delta}_\alpha$
- $\pi^{\nu_\delta,\nu}_\alpha(\kappa^{\nu_\delta}_\alpha) = \kappa^\nu_\alpha$
- $\pi^{\nu_\delta,\nu}_\alpha(b^{\nu_\delta}_\alpha) = b^\nu_\alpha$

Then $G^b_{\nu_\delta,\nu}$ is an element of U since the system $\mathcal{S}_b$ lines up on R''_b. By Lemma 5 there are U-many α such that $\alpha \in G_\delta$ for cofinally many $\delta < \alpha$. □

If we make the additional assumption on U that $|^{\omega_2}\omega_1/U| = \omega_2$, then *every* system of objects from the winning mouse lines up on a stationary set:

Lemma 24. *Assume U is a weakly normal ultrafilter concentrating on S^2_1 and $|^{\omega_2}\omega_1/U| = \omega_2$. Then for every system $\mathcal{S} = \langle Obj^\nu_{(-)} | \nu \in R\rangle$ of objects from the winning mouse, there is a stationary $R'' \subset R$ on which $\mathcal{S}$ lines up.*

Proof. Let R' and $\bar\tau < \omega_3$ be the stationary set and ordinal, respectively, guaranteed by Lemma 15. For each $\nu \in R'$ define

$$\bar\phi^\nu(\alpha) = \text{the preimage of } \mathrm{Obj}^\nu_\alpha \text{ by the iteration map } \pi^{\bar\tau,\nu}_\alpha.$$

Recall by Lemma 15 that for every $\nu \in R'$, there are U-many α such that the level of the mouse $N^{\bar\tau}_\alpha$ at which $\bar\phi^\nu(\alpha)$ appears is strictly less than $h_{\bar\tau}(\alpha)$. For each transitive $N \in H_{\omega_2}$ fix an injection $g_N : N \to \omega_1$. Let $N(\bar\tau, \alpha)$ be the initial segment of the mouse $N^{\bar\tau}_\alpha$ corresponding to level $h_{\bar\tau}(\alpha)$. Define

$$\psi^\nu(\alpha) :\simeq g_{N(\bar\tau,\alpha)}(\bar\phi^\nu(\alpha))$$

Note $\psi^\nu(\alpha)$ is defined for U-many α; so each ψ^ν is a function whose domain is in U and maps into ω_1. Since $|^{\omega_2}\omega_1/U| = \omega_2$ and R' is a stationary subset of ω_3, then by the ω_3-completeness of NS_{ω_3} there is a stationary $R'' \subset R'$ such that $\psi^{\bar\nu} =_U \psi^\nu$ for every $\bar\nu, \nu$ in R''. It follows that $\bar\phi^{\bar\nu} =_U \bar\phi^\nu$ for every $\bar\nu, \nu$ in R'', and so $\mathcal{S}$ lines up on R''. □

For the rest of the paper, D' will refer to the collection of $\nu \in D \cap S^3_2$ such that $\kappa^\nu_{(-)} =_U h_\nu$; by Corollary 20 D' is almost all of S^3_2.

5. Proof of part 1 of Theorem 1

As in Chapter 8 of [14], for mice below 0-pistol, $o^M_*(\kappa)$ denotes the "Mitchell order of κ in M"; more precisely, the ordertype of the collection of $\nu \geq \kappa^{+M}$ which index an extender with critical point κ. $o^M(\kappa)$ denotes the least primitively recursively closed ordinal which is at least κ^+ and does not index an extender with critical point κ. If E is an extender on the M sequence which has critical point κ and is generated by a single normal measure of M-Mitchell order λ, we will use $\mathcal{U}^M(\kappa, \lambda)$ to denote this normal measure. For the rest of the paper, our background assumption is that there are no extenders on any mice with two generators; so *every* extender is generated by such a normal measure.

Theorem 25. *Let $b \in \wp^K(\omega_3)$ code a wellordering of ω_3. Then $o^K_*(\nu) > otp(b \cap \nu)$ for all but nonstationarily many $\nu \in S^3_2$ ("all but nonstationarily many" is in the sense of V).*

This could be equivalently formulated in terms of canonical functions on ω_3, but we will avoid that formulation to avoid confusion with the canonical functions on ω_2 which are already in use. We also find a nice characterization of the measures on such ν (Lemma 28). This characterization will then be used to define a K-ultrafilter on ω_3 and show it has the desired properties.[8]

Let $\gamma := \mathrm{otp}(b)$; so $\gamma < \omega_3^{+K}$. We assume by induction that Theorem 25 holds for all ordinals $< \gamma$; i.e. whenever $\tau < \gamma = \mathrm{otp}(b)$ then there is an ω_2-club $C_{>\tau} \subset \omega_3$ in V such that $o^K_*(\nu) > \mathrm{otp}(b_\tau \cap \nu)$ for every $\nu \in C_{>\tau}$, where $b_\tau \in \wp^K(\omega_3)$ has ordertype τ.

Claim 26. *There is an ω_2-club $C_{\geq b} \subset \omega_3$ such that $o^K_*(\nu) \geq otp(b \cap \nu)$ for every $\nu \in C_{\geq b}$.*

Proof. Fix some sequence $\langle \tau_i | i < \mathrm{cf}(\gamma)\rangle \in V$ cofinal in γ; if γ is a successor ordinal, say $\gamma = \tau_0 + 1$, the sequence is just $\langle \tau_0 \rangle$. Fix a large regular $\theta > 2^{\omega_3}$ and consider the collection $\tilde{C}$ of all $Z \in P_{\omega_3}(H_\theta)$ such that $Z \prec (H_\theta, \langle (\tau_i, C_{>\tau_i}, b_{\tau_i}) | i < \mathrm{cf}(\gamma)\rangle, ...)$ and $Z \cap \omega_3 \in S^3_2$. Let $C_{\geq b} := \{Z \cap \omega_3 | Z \in \tilde{C}\}$. For $Z \in \tilde{C}$ let $\pi_Z : H_Z \to H_\theta$ be the inverse of the Mostowski collapse of Z. Let $\nu_Z := cr(\pi_Z) = Z \cap \omega_3$.

Now for all $i \in Z \cap \omega_3 = \nu_Z$, $\nu_Z \in C_{>\tau_i}$.[9] So:

- if $\mathrm{cf}(\gamma) = \omega_3$, then $\nu_Z \in \triangle_{i<\omega_3} C_{>\tau_i}$ and $\mathrm{otp}(b \cap \nu_Z) = \sup_{i<\nu_Z} \mathrm{otp}(b_{\tau_i} \cap \nu_Z)$.
- if $\mathrm{cf}(\gamma) < \omega_3$ is a limit ordinal, then $\nu_Z \in \bigcap_{i<\mathrm{cf}(\gamma)} C_{>\tau_i}$ and $\mathrm{otp}(b \cap \nu_Z) = \sup_{i<\mathrm{cf}(\gamma)} \mathrm{otp}(b_{\tau_i} \cap \nu_Z)$
- if γ is a successor ordinal, say $\gamma = \tau + 1$, then $\nu_Z \in C_{>\tau}$ and $\mathrm{otp}(b \cap \nu_Z) = \mathrm{otp}(b_\tau \cap \nu_Z) + 1$.

In all 3 cases, $o^K_*(\nu_Z) \geq otp(b \cap \nu_Z)$. □

Let $b^\nu := b \cap \nu$ for every $\nu < \omega_3$. Note that if $\nu \in D' \cap C_{\geq b}$, then $h_\nu(\alpha) = \kappa^\nu_\alpha$, $\mu^\nu_\alpha = o^{K^\nu_\alpha}(\kappa^\nu_\alpha)$,[10] and the Mitchell order of $E^{\bar{N}^\nu_\alpha}_{\mu^\nu_\alpha}$ is at least $\mathrm{otp}(b^\nu_\alpha)$ for U-many α. Furthermore, by Corollary 16 there is no

[8] One of these properties is that the Mitchell order of this K-ultrafilter will be $otp(b)$. Since b was an arbitrary wellorder of ω_3 in K, this would imply that $o^K(\kappa) \geq \kappa^{+K}$, where $\kappa = \omega_3^V$.

[9] because $C_{>\tau_i} \in Z$ and so (by elementarity) ν_Z is a limit of $C_{>\tau_i}$; since ν_Z has cofinality ω_2, and $C_{>\tau_i}$ is an ω_2-club, then $\nu_Z \in C_{>\tau_i}$.

[10] This follows from (5).

truncation at stage θ^ν_α, so the extender applied by the K side at this stage is total in N^ν_α. In particular:

(24) For every $\nu \in C_{\geq b}$ there are U-many α such that N^ν_α has a total extender on $h_\nu(\alpha)$ of Mitchell order $\mathrm{otp}(b^\nu_\alpha)$.

So for every $\nu \in C_{\geq b}$ the following definition makes sense: define an ultrafilter W^{b^ν} on $\wp^K(\nu)$ by

$$z \in W^{b^\nu} \text{ iff } \{\alpha \in \bar{S}^\nu | z^\nu_\alpha \in \mathcal{U}^{N^\nu_\alpha}(h_\nu(\alpha), \mathrm{otp}(b^\nu_\alpha))\} \in U.$$

(i.e. iff $ult(V,U) \models [z^\nu]_U \in \mathcal{U}^{[N^\nu]_U}([h_\nu]_U, [b^\nu_{(-)}]_U)$.) Here b^ν_α and z^ν_α are defined as $(\sigma^\nu_\alpha)^{-1}(b)$ and $(\sigma^\nu_\alpha)^{-1}(z)$ (respectively), for those α such that $b, z \in \mathrm{rng}(\sigma^\nu_\alpha)$ (see Definition 2). For the remainder of the proof, to cut down on notation we will often just write b^ν_α instead of $\mathrm{otp}(b^\nu_\alpha)$; e.g. $\mathcal{U}^{N^\nu_\alpha}(\kappa^\nu_\alpha, b^\nu_\alpha)$ means the normal measure of Mitchell order $\mathrm{otp}(b^\nu_\alpha)$.

It is clear that W^{b^ν} is an ultrafilter on $\wp^K(\nu)$. Our goals are to ultimately show that W^{b^ν} generates K's extender of order $\mathrm{otp}(b^\nu)$, *and* to characterize the W^{b^ν}'s in a way that allows us to build a K-measure on ω_3 of Mitchell order $\mathrm{otp}(b)$.

Lemma 27. *For all but nonstationarily many $\nu \in S^3_2$: $o^K_*(\nu) > otp(b^\nu)$.*

Proof. Suppose this fails; then there is a stationary $R_b \subset C_{\geq b}$ such that $o^K_*(\nu) = \mathrm{otp}(b^\nu)$ for every $\nu \in R_b$.

Since $b^\nu \in \wp^K(\nu)$, then by Corollary 21, $\mathcal{S}_b := \langle (b^\nu_{(-)}, \kappa^\nu_{(-)}) | \nu \in R_b\rangle$ is a system of objects from the winning mouse. Since the b^ν are initial segments of b, by Corollary 22 there is a stationary $R''_b \subset R_b$ on which $\mathcal{S}_b$ lines up.

For the remainder of the proof of Lemma 27, fix some $\nu \in R''_b \cap \mathrm{Lim}(R''_b)$ and any $t_\nu = \langle \nu_\delta | \delta < \omega_2\rangle$ which is a subset of R''_b and cofinal in ν.

Claim 27.1. *$\langle \nu_\delta | \delta < \omega_2\rangle$ generates W^{b^ν}; i.e. $z \in W^{b^\nu}$ iff $\nu_\delta \in z$ for sufficiently large $\delta < \omega_2$.*

(Note we are still proving Lemma 27 by contradiction; the current claim will not *in general characterize the measures W^{b^ν}.)*

Proof. (of Claim 27.1) Let $z \in W^{b^\nu}$; so $z_{(-)} \in_U \mathcal{U}^{N^\nu_{(-)}}(h_\nu(-), b^\nu_{(-)})$ by definition. Recall $\theta^\nu_\alpha = \sup_{\delta<\alpha} \theta^{\nu_\delta}_\alpha$ is a limit ordinal for U-many α. So for such α there is a $\delta(\alpha) < \alpha$ such that a thread to z_α appears by stage $\theta^{\nu_{\delta(\alpha)}}_\alpha$ of the K vs. K^ν_α coiteration. By weak normality of U, there is a $\hat{\delta}$ such that for U-many α, a thread to z_α appears by stage $\theta^{\nu_{\hat{\delta}}}_\alpha$. So:

(25) For every $\delta \in [\hat{\delta}, \omega_2)$: $z_{(-)} \in_U \mathrm{range}(\pi^{\nu_\delta,\nu}_{(-)})$.

We will show $\nu_\delta \in z$ for every $\delta \in [\hat{\delta}, \omega_2)$: Fix such a δ. Recall that $\kappa^{\nu_\delta}_{(-)} =_U h_{\nu_\delta}(-)$ and $\kappa^{\nu}_{(-)} =_U h_\nu(-)$. ν_δ and ν are both elements of R''_b and $\mathcal{S}_b$ lines up on R''_b; so $\pi^{\nu_\delta,\nu}_\alpha(b^{\nu_\delta}_\alpha, \kappa^{\nu_\delta}_\alpha) = (b^\nu_\alpha, \kappa^\nu_\alpha)$ for U-many α. And since $z_{(-)} \in_U \mathcal{U}^{N^\nu_{(-)}}(h_\nu(-), b^\nu_{(-)})$ (by assumption), then for U-many α:

(26) $$z_\alpha \cap \kappa^{\nu_\delta}_\alpha = (\pi^{\nu_\delta,\nu}_\alpha)^{-1}(z_\alpha) \in \mathcal{U}^{N^{\nu_\delta}_\alpha}(\kappa^{\nu_\delta}_\alpha, b^\nu_\alpha \cap \kappa^{\nu_\delta}_\alpha) = \mathcal{U}^{N^{\nu_\delta}_\alpha}(\kappa^{\nu_\delta}_\alpha, b^{\nu_\delta}_\alpha)$$

Since ν_δ is in R''_b then

(27) $$\begin{aligned} b^{\nu_\delta}_{(-)} &=_U \text{the Mitchell order of } h_{\nu_\delta}(-) \text{ in } K^{\nu_\delta}_{(-)} \\ &=_U \text{the Mitchell order of } \kappa^{\nu_\delta}_{(-)} \text{ in } K^{\nu_\delta}_{(-)} \end{aligned}$$

So by (26) and (27), there are U-many α such that $z_\alpha \cap h_{\nu_\delta}(\alpha)$ is an element of the measure applied at stage $\theta^{\nu_\delta}_\alpha$—and so $h_{\nu_\delta}(\alpha) \in z_\alpha$—for U-many α. Pick one such α; then $\sigma^\nu_\alpha(h_{\nu_\delta}(\alpha)) = \nu_\delta$ is an element of $\sigma^\nu_\alpha(z_\alpha) = z$.

This completes the proof that every element of W^{b^ν} contains a tail end of the set $\{\nu_\delta | \delta < \omega_2\}$. Since W^{b^ν} is an ultrafilter on $\wp^K(\nu)$, then the converse is also true. □(Claim 27.1)

We will complete the proof of Lemma 27 by showing W^{b^ν} is on K's sequence and has Mitchell order $\mathrm{otp}(b^\nu)$; this will contradict that $\nu \in R_b$.

Claim 27.2. $\{\xi < \nu | o^K_*(\xi) = otp(b \cap \xi)\} \in W^{b^\nu}$

Proof. This follows easily from the definition of W^{b^ν} and the fact that each $\mathcal{U}^{N^\nu_\alpha}(h_\nu(\alpha), b^\nu_\alpha)$ has Mitchell order b^ν_α. □

By Corollary 29 from [2], to see that W^{b^ν} generates an extender on K's extender sequence, it suffices[11] to show that W^{b^ν} is normal with respect to K and that $ult(K|\nu^{+K}, W^{b^\nu})$ is wellfounded. Let $G^b_{s_\nu} \in U$ be the set from the conclusion of Corollary 23 (recall $s_\nu = \langle \nu_\delta | \delta < \omega_2 \rangle$ is some sequence which is contained in R''_b and cofinal in ν). Pick an $X \in P_{\omega_2}(H_\theta)$ with $W^{b^\nu} \in X \prec H_\theta$ and $\alpha_X := X \cap \omega_2 \in G^b_{s_\nu}$; there is such an X because $G^b_{s_\nu} \in U$ and is thus a stationary subset of S^2_1 . Let $\sigma_X : H_X \to H_\theta$ be the inverse of the Mostowski collapsing map. Let $W_X := \sigma^{-1}_X(W^{b^\nu})$. By Claim 27.1:

(28) $$W_X \text{ is generated by } \sigma^{-1}_X(\langle \nu_\delta | \delta < \omega_2 \rangle) = \langle h_{\nu_\delta}(\alpha_X) | \delta < \alpha_X \rangle$$

[11]Under the assumption that 0-pistol does not exist, which we are assuming throughout.

Note that $o_*^K(\nu_\delta) = b^{\nu_\delta}$ for every $\delta < \omega_2$, and so:

(29) $$o_*^{K_X}(h_{\nu_\delta}(\alpha_X)) = b^{\nu_\delta}_{\alpha_X} \text{ for every } \delta < \alpha_X.$$

Let I be the collection of $\delta < \alpha_X$ such that (23) holds; since $\alpha_X \in G^b_{s_X}$ then I is cofinal in α_X. Then the definition of I and (29) imply that the Mitchell order of the extender applied at stage $\theta^{\nu_\delta}_{\alpha_X}$ is equal to $\mathrm{otp}(b^{\nu_\delta}_{\alpha_X})$, and $\pi^{\nu_\delta,\nu}_{\alpha_X}(b^{\nu_\delta}_{\alpha_X}) = b^\nu_{\alpha_X}$ for every $\delta \in I$. I.e. the measure $\mathcal{U}^\nu_{\alpha_X} := \mathcal{U}^{N^\nu_{\alpha_X}}(h_\nu(\alpha_X), b^\nu_{\alpha_X})$ in the iterate $N^\nu_{\alpha_X}$ is the result of "repeating a measure" at stages of the form $\theta^{\nu_\delta}_{\alpha_X}$ for $\delta \in I$; and the critical point at each such stage was of the form $h_{\nu_\delta}(\alpha_X)$. So

(30) The measure $\mathcal{U}^\nu_{\alpha_X}$ is generated by $\langle h_{\nu_\delta}(\alpha_X) | \delta \in I\rangle$.

Since $\wp^{K_X}(h_\nu(\alpha_X)) \subseteq \wp^{N^\nu_{\alpha_X}}(h_\nu(\alpha_X))$, then (28) and (30) imply:

(31) $$W_X = \mathcal{U}^\nu_{\alpha_X} \cap \wp^{K_X}(h_\nu(\alpha_X))$$

(In fact they're equal; we could WLOG assume there is no truncation at stage $\theta^\nu_{\alpha_X}$ by Corollary 16.)

Now suppose that *either*

- W^{b^ν} were *not* normal with respect to K, or
- $ult(K|\nu^{+K}, W^{b^\nu})$ were illfounded.

Then H_X would believe the same about W_X with respect to K_X. We show how to achieve a contradiction if $ult(K|\nu^{+K}, W^{b^\nu})$ is illfounded; if W^{b^ν} fails to be normal with respect to K, the proof is similar. So suppose $ult(K|\nu^{+K}, W^{b^\nu})$ is illfounded. Then by elementarity of σ_X, $ult(K_X|h_\nu(\alpha_X)^{+K_X}, W_X)$ is illfounded; let $\{f_n | n \in \omega\} \in H_X$ witness this fact. Since $h_\nu(\alpha_X)$ is the largest cardinal in $K_X|h_\nu(\alpha_X)^{+K_X}$, then WLOG we assume $f_n : h_\nu(\alpha_X) \to h_\nu(\alpha_X)$. But $h_\nu(\alpha_X) = \kappa^\nu_{\alpha_X}$ and so $\wp^{K_X}(h_\nu(\alpha_X)) \subseteq \wp^{N^\nu_{\alpha_X}}(h_\nu(\alpha_X))$. By (31), $\{f_n|n \in \omega\}$ witnesses an illfounded chain in $ult(N^\nu_{\alpha_X}, \mathcal{U}^\nu_{\alpha_X})$, which is a contradiction because $\mathcal{U}^\nu_{\alpha_X}$ is on the extender sequence of the mouse $N^\nu_{\alpha_X}$. (In the case where we assume W_X fails to be normal in K_X, we would contradict the fact that $\mathcal{U}^\nu_{\alpha_X}$ is normal with respect to $N^\nu_{\alpha_X}$.) $\square$(Lemma 27)

So K *has* an extender of order $\mathrm{otp}(b \cap \nu)$ for all but nonstationarily many $\nu \in S^3_2$. This concludes the proof of Theorem 25. Next we show that, for many ν, the extender on ν of order b^ν in K is exactly (the extender generated by) W^{b^ν}. This fact will be used later in the definition of measures on ω_3.

Lemma 28. *Let $C_{>b}$ be an ω_2-closed unbounded subset of ν such that $o^K_*(\nu) > otp(b \cap \nu)$ for every $\nu \in C_{>b}$ (such a set exists by Lemma 27). Let R_b be any stationary subset of $C_{>b}$. Then there is a stationary $R''_b \subset R_b$ with the properties:*

- *W^{b^ν} is (generates) K's extender with critical point ν of order b^ν for every $\nu \in R''_b$*
- *Whenever $\nu \in R''_b \cap Lim(R''_b)$ then W^{b^ν} has the following characterization:*

$$z \in W^{b^\nu} \textit{ iff } z \cap \nu' \in \mathcal{U}^K(\nu', b^{\nu'}) \textit{ for sufficiently large } \nu' \in \nu \cap R''_b. \tag{32}$$

Proof. Let $\mathcal{S} = \langle b^\nu_{(-)}, \kappa^\nu_{(-)} | \nu \in R_b \rangle$; by Corollary 21 $\mathcal{S}$ is a system of objects of the winning mouse. By Corollary 22 there is a stationary $R''_b \subset R_b$ on which $\mathcal{S}$ lines up.

Pick any $\nu \in R''_b \cap \mathrm{Lim}(R''_b)$ and fix any sequence $t_\nu = \langle \nu_\delta | \delta < \omega_2 \rangle$ of points in R''_b which is cofinal in ν. So for each $\delta < \omega_2$ we know K *has* a measure of order b^{ν_δ} on ν_δ, but we don't yet know that it is just $W^{b^{\nu_\delta}}$. Similarly we know K has a measure of order b^ν on ν but do not yet know that it is W^{b^ν}.

The proof of (32) is very similar to the proof of Claim 27.1, but note that in the current proof W^{b^ν} is being characterized by *reflection* rather than by a generating sequence. Very briefly, suppose we are given some $z \in W^{b^\nu}$. Use weak normality of U and the fact that θ^ν_α is a limit ordinal for U-many α to find a $\delta^* < \omega_2$ such that $z_{(-)} \in_U \mathrm{range}(\pi^{\nu_{\delta^*},\nu}_{(-)})$ (and for U-many α there are no truncations at stages in the interval $[\theta^{\nu_{\delta^*}}_\alpha, \theta^\nu_\alpha)$). Then pick any $\delta \in [\delta^*, \omega_2)$. First, since $\nu_\delta \in R_b$ then $o^K_*(\nu_\delta) > b^{\nu_\delta}$; this implies that for U-many α, $K^{\nu_\delta}_\alpha$ and $N^{\nu_\delta}_\alpha$ have exactly the same measure of order $b^{\nu_\delta}_\alpha$ on $h_{\nu_\delta}(\alpha)$, since the index of the measure applied on the $N^{\nu_\delta}_\alpha$ side is just the least ordinal which does not index an extender on the $K^{\nu_\delta}_\alpha$ side (see (5)). Moreover, by coherency of the iteration, $N^{\nu_\delta}_\alpha$ and N^ν_α have the same measure of order $b^{\nu_\delta}_\alpha$ on $h_{\nu_\delta}(\alpha)$. Finally, since the $K^{\nu_\delta}_\alpha$ side of the coiteration is trivial, then $K^{\nu_\delta}_\alpha$ is just an initial segment of K^ν_α. To summarize: the mice N^ν_α, $N^{\nu_\delta}_\alpha$, and K^ν_α all have exactly the same measure of order $b^{\nu_\delta}_\alpha$ on $h_{\nu_\delta}(\alpha)$; let $\mathcal{V}_\delta$ denote this common measure. Then use the fact that ν_δ, ν are both in R''_b, along with the assumption that $z \in W^{b^\nu}$, to show that $z_\alpha \cap h_{\nu_\delta}(\alpha) \in \mathcal{V}_\delta$; then applying σ^ν_α yields that $z \cap \nu_\delta \in \mathcal{U}^K(\nu_\delta, b^{\nu_\delta})$. This completes the sketch of the proof of (32).

Finally, we use (32) to show that W^{b^ν} is in fact on K's extender sequence. The proof is similar to the end of the proof of Lemma 27. Let $G^b_{t_\nu} \in U$ be the set from the conclusion of Corollary 23. Pick an $X \in P_{\omega_2}(H_\theta)$ of cardinality ω_1 such that $t_\nu \in X$ and $\alpha_X \in G^b_{t_\nu}$; again, this is possible because $G^b_{t_\nu} \in U$ and is thus stationary. Let $\sigma_X : H_X \to H_\theta$ be the inverse of the Mostowski collapsing map of X, and let $W_X = \sigma_X^{-1}(W^{b^\nu})$. Let I be the collection of $\delta < \alpha_X$ where (23) holds; since $\alpha_X \in G^b_{s_\nu}$ then I is cofinal in α_X. Now for every $\delta \in I$, the measure on $h_{\nu_\delta}(\alpha_X)$ of Mitchell order $b^{\nu_\delta}_{\alpha_X}$ is exactly the same in the mice $K_X, N^{\nu_\delta}_{\alpha_X}$, and $N^{\nu}_{\alpha_X}$; let $\mathcal{V}_\delta$ denote this common measure. Then the measure $\mathcal{U}^{\nu}_{\alpha_X} := \mathcal{U}^{N^{\nu}_{\alpha_X}}(h_\nu(\alpha_X), b^{\nu}_{\alpha_X})$ has the following characterization:

$$z \in \mathcal{U}^{\nu}_{\alpha_X} \iff z \cap h_{\nu_\delta}(\alpha_X) \in \mathcal{V}_\delta \text{ for all sufficiently large } \delta \in I. \tag{33}$$

The elementarity of σ_X and (32) imply:

$$z \in W_X \iff z \cap h_{\nu_\delta}(\alpha_X) \in \mathcal{V}_\delta \text{ for sufficiently large } \delta < \alpha_X. \tag{34}$$

Then (33) and (34) imply

$$W_X = \mathcal{U}^{\nu}_{\alpha_X} \cap \wp^{K_X}(h_\nu(\alpha_X)) \tag{35}$$

Similarly to the proof of Lemma 27, (35) can be used to show that W_X is normal with respect to K_X and that $ult(K_X, W_X)$ is wellfounded. So W_X generates an extender on K_X's extender sequence, and so by elementarity of σ_X, W^{b^ν} generates an extender on K's extender sequence (and (32) provided the desired characterization of W^{b^ν}).

□(Lemma 28)

Finally, we build the measures on ω_3. Fix a stationary R''_b as in the conclusion of Lemma 28. Although we will not use this fact, it is interesting to note that such an R''_b can be obtained within *any* stationary subset of S^3_2 (see the statement of Lemma 28).

Define a filter F^b on $\wp^K(\omega_3)$ by:

$$z \in F^b \text{ iff } z \cap \nu \in W^{b^\nu} \text{ for sufficiently large } \nu \in R''_b$$

Claim 29. *F^b is an ultrafilter on $\wp^K(\omega_3)$.*

Proof. Suppose not; so there is a $z \in \wp^K(\omega_3)$ such that both $R''_{b,z} := \{\nu \in R''_b | z \cap \nu \in W^{b^\nu}\}$ and $R''_{b,z^c} := \{\nu \in R''_b | z^c \cap \nu \in W^{b^\nu}\}$ are cofinal in ω_3. At least one of $R''_{b,z}, R''_{b,z^c}$ must be stationary; WLOG assume $R''_{b,z}$ is stationary. Then since R''_{b,z^c} is unbounded in ω_3, there is a $\nu \in R''_{b,z} \cap \mathrm{Lim}(R''_{b,z^c})$. Fix a sequence $t_\nu = \langle \nu_\delta | \delta < \omega_2 \rangle$ of points in R''_{b,z^c} which is cofinal in ν. Since $z \cap \nu \in W^{\nu^b}$ and $t_\nu \subset R''_b$, then Lemma 28 yields that $z \cap \nu_\delta \in W^{b^{\nu_\delta}}$ for sufficiently large $\delta < \omega_2$. But this contradicts that each ν_δ is an element of R''_{b,z^c}. □

Using the definition of F^b (and the fact that each W^{b^ν} is on K's extender sequence), it is easy to see that F^b is normal with respect to K, $ult(K, F^b)$ is wellfounded, and F^b concentrates on the set $\{\xi < \omega_3 | o^K_*(\xi) = \mathrm{otp}(b \cap \xi)\}$. By Corollary 29 from [2], F^b generates K's total extender on ω_3 of Mitchell order $\mathrm{otp}(b)$.

6. Proof of part 2 of Theorem 1

In this section we make the additional assumption that $|{}^{\omega_2}\omega_1/U| = \omega_2$ in order to show there is a mouse which has an extender with 2 generators. We keep all notation from the previous section; in particular D' is the set of $\nu \in S^3_2$ such that $h_\nu =_U \kappa^\nu_{(-)}$; recall D' is almost all of S^3_2.

For $\nu \in D'$, by Lemma 13 there are U-many α such that $\theta^\nu_\alpha + 2 \leq \theta^{\nu^*}_\alpha$ and θ^ν_α is a limit ordinal, among other properties; recall ν^* is the least element of D above ν. For such α, let E^ν_α denote the extender applied at stage θ^ν_α; recall μ^ν_α denotes the index of E^ν_α on N^ν_α's extender sequence, and $\mu^\nu_\alpha = o^{K^\nu_\alpha}(h_\nu(\alpha))$ (for U-many α). For an ordinal $\eta < \mu^\nu_\alpha = lh(E^\nu_\alpha)$, $(E^\nu_\alpha)_\eta$ denotes the N^ν_α-ultrafilter $\{z | \eta \in E^\nu_\alpha(z)\}$ (i.e. viewing E^ν_α as a hypermeasure, it is the η-th element of the hypermeasure). If η is primitive recursively closed then $E^\nu_\alpha | \eta$ denotes the extender $z \mapsto E^\nu_\alpha(z) \cap \eta$.

For each $\nu \in D'$ and $\eta < o^K(\nu)$ define a K-ultrafilter $W_{\nu,\eta}$ by:

$$z \in W_{\nu,\eta} \text{ iff } z \in \wp^K(\nu) \text{ and } \{\alpha | z_\alpha \in (E^\nu_\alpha)_{\eta_\alpha}\} \in U. \tag{36}$$

Let $G^\nu := \langle W_{\nu,\eta} | \eta < o^K(\nu)\rangle$. G^ν is a sequence of ultrafilters, though at the moment it is not clear that G^ν is an extender. Note by the previous section, K has total extenders on ω_3^V, so $o^K(\nu) < \omega_3$ for each $\nu < \omega_3$ (since we are assuming 0-pistol does not exist; i.e. there are no overlapping extenders).

Consider any $\nu \in D'$ and any $X \in S^\nu$ such that there is no truncation at stage $\theta^\nu_{\alpha_X}$ (recall this holds for U-many α). Let $\pi_X : H_X \to H_\theta$ be the inverse of the Mostowski collapsing map of X. Since E^ν_α is an extender on the mouse N^ν_α and N^ν_α, K_X agree below their common successor of $cr(E^\nu_{\alpha_X})$, then $E^\nu_{\alpha_X}$ is K_X-correct but is not on the K_X sequence; this implies:

$$E^\nu_{\alpha_X} \text{ is not an element of } H_X. \tag{37}$$

For each $\nu \in D'$ define a function s^ν by:

$$s^\nu(\alpha) :\simeq \text{ the least } \zeta < o^{K^\nu_\alpha}(h_\nu(\alpha)) \text{ such that } G^\nu_\alpha(\zeta) \neq (E^\nu_\alpha)_\zeta. \tag{38}$$

By (37), $s^\nu(\alpha)$ exists for U-many α and $s^\nu(\alpha) < h_{o^K(\nu)}(\alpha)$ (for each $\nu \in D'$). Since each ultrafilter $(E^\nu_\alpha)_\xi$ for $\xi < h_\nu(\alpha) = cr(E^\nu_\alpha)$ is simply the principal ultrafilter generated by ξ, it is easy to see that $G^\nu_\alpha(\xi) = (E^\nu_\alpha)_\xi$ for each such ξ. So $s^\nu(\alpha) \geq h_\nu(\alpha)$. To summarize (note we have not yet used the assumption $|{}^{\omega_2}\omega_1/U| = \omega_2$):

$$h_\nu \leq_U s^\nu <_U h_{o^K(\nu)} \text{ for every } \nu \in D'. \tag{39}$$

Claim 30. *For all but nonstationarily many $\nu \in S^3_2$: s^ν is* not *U-equivalent to any canonical function of the form h_η (for some $\eta < \omega_3$).*

Proof. Suppose to the contrary that there is a stationary $R \subset D'$ and for each $\nu \in R$, there is an η^ν such that $s_\nu =_U \eta^\nu_{(-)}$ $(= h_{\eta^\nu})$. By (39), we know $\eta^\nu \in [\nu, o^K(\nu))$.

Consider the system $\mathcal{S} := \langle(\mu^\nu_{(-)}, s^\nu(-)|\nu \in R\rangle$ of objects from the winning mouse. By Lemma 24 there is a stationary $R'' \subset R$ on which $\mathcal{S}$ lines up. Fix a $\nu \in R'' \cap \mathrm{Lim}(R'')$. Then:

(40) W_{ν,η^ν} is generated by $\langle \eta^{\nu'}|\nu' \in R'' \cap \nu\rangle$ (i.e. $z \in W_{\nu,\eta^\nu}$ iff z contains a tail of that sequence).

The proof of (40) is similar to the proof of Claim 27.1. Consider any sequence $\langle \nu_\delta|\delta < \omega_2\rangle$ of points in $R'' \cap \nu$ which is cofinal in ν. Pick any $z \in \wp^K(\nu)$. For each α pick a $\delta(\alpha)$ so that a thread to $(z)_\alpha$ appears by stage $\theta^{\nu_{\delta(\alpha)}}_\alpha$ (in the K vs. K^ν_α coiteration). By weak normality of U there is a $\delta^* < \omega_2$ and U-many α such that $\delta(\alpha) \leq \delta^*$. Then show that $\eta^{\nu_\delta} \in z$ for every $\delta \in [\delta^*, \omega_2)$; the proof is almost identical to the proof of Claim 27.1. The only difference is that here we use that x is an element of $(E^{\nu_\delta}_\alpha)_{h_{\eta^{\nu_\delta}}(\alpha)}$ iff $h_{\eta^{\nu_\delta}}(\alpha) \in \pi^{\nu_\delta,\nu}_\alpha(x)$.

Since ν and all the ν_δ are elements of R'' and $\mathcal{S}$ lines up on R'', then for each $\delta < \omega_2$ there is a set $A_{\nu_\delta,\nu} \in U$ such that $\pi^{\nu_\delta,\nu}_\alpha((\mu^{\nu_\delta}_\alpha, \eta^{\nu_\delta}_\alpha)) = (\mu^\nu_\alpha, \eta^\nu_\alpha)$ for every $\alpha \in A_{\nu_\delta,\nu}$. Also by assumption there is a set $B_{\nu_\delta,\nu} \in U$ so that $h_{\eta^{\nu_\delta}}(\alpha) = s^{\nu_\delta}(\alpha)$ and $h_{\eta^\nu}(\alpha) = s^\nu(\alpha)$ for every $\alpha \in B_{\nu_\delta,\nu}$. Let $C_\delta := A_{\nu_\delta,\nu} \cap B_{\nu_\delta,\nu}$. By Lemma 5 there are U-many α such that $I_\alpha = \{\delta < \alpha | \alpha \in C_\delta\}$ is cofinal in α; let $C \in U$ be the collection of such α. Then for every $\alpha \in C$: $P_\alpha := \langle h_{\eta^{\nu_\delta}}(\alpha)|\delta \in I_\alpha\rangle$ is a generating sequence for the ultrafilter $(E^\nu_\alpha)_{h_{\eta^\nu}(\alpha)}$.

Now pick any $X \prec (H_\theta, \in, \Delta, ...)$ such that $t_\nu \in X$ and $\alpha = X \cap \omega_2$ is an element of C; this is possible because $C \in U$ and is thus stationary. By (40) and elementarity of π_X, $Q_X := \pi^{-1}_X(\langle \eta^{\nu_\delta}|\delta < \omega_2\rangle) = \langle h_{\eta^{\nu_\delta}}(\alpha_X)|\delta < \alpha_X\rangle$ and this sequence generates the ultrafilter $\pi^{-1}_X(W_{\nu,\eta^\nu}) = (\pi^{-1}_X(G^\nu))(h_{\eta^\nu}(\alpha_X))$.

But P_{α_X} is a cofinal subset of Q_X. Since P_{α_X} generates $(E^\nu_{\alpha_X})_{h_{\eta^\nu}(\alpha_X)}$ and Q_X generates $(\pi^{-1}_X(G^\nu))(h_{\eta^\nu}(\alpha_X))$, then these must be the same ultrafilter. But $h_{\eta^\nu}(\alpha_X) = s^\nu(\alpha_X)$ (since $\alpha_X \in C$). This contradicts the definition of the function s^ν in (38). $\square$

Claim 30, along with (39), imply that many of the mice in the coiterations of this paper have extenders with at least two generators (which is a bit stronger than $o(\kappa) = \kappa^{++}$). To see this, pick any $X \prec (H_\theta, \in, \Delta, ...)$ such that $h_\nu(\alpha) < s^\nu(\alpha)$, where $\alpha = X \cap \omega_2$. So in particular $(E^\nu_\alpha)_{h_\nu(\alpha)}$ is an element of H_X and if E^ν_α had no other generator, then the entire extender E^ν_α would be an element of H_X, which contradicts (37).

We also point out the following fact: let $\tau^\nu < \omega_3$ be minimal such that $s^\nu \leq_U h_{\tau^\nu}$; by Claim 30, this inequality is in fact strict. Since $s^\nu >_U h_\xi$ for every $\xi < \tau^\nu$ and $s^\nu <_U h_{\tau^\nu}$, then by Lemma 9 the cofinality of τ^ν must be strictly less than ω_2.

References

[1] C. C. Chang and H. J. Keisler, *Model theory*, 3rd ed., Studies in Logic and the Foundations of Mathematics, vol. 73, North-Holland Publishing Co., Amsterdam, 1990. MR1059055 (91c:03026)

[2] Sean Cox, *Covering theorems for the core model, and an application to stationary set reflection*, Ann. Pure Appl. Logic **161** (2009), no. 1, 66–93, DOI 10.1016/j.apal.2009.06.001. MR2567927

[3] ———, Ph.D. dissertation, University of California, Irvine, 2009.

[4] James Cummings, Matthew Foreman, and Menachem Magidor, *Canonical structure in the universe of set theory. I*, Ann. Pure Appl. Logic **129** (2004), no. 1-3, 211–243. MR2078366 (2005k:03105)

[5] D. Donder, R. B. Jensen, and B. J. Koppelberg, *Some applications of the core model*, Set theory and model theory (Bonn, 1979), Lecture Notes in Math., vol. 872, Springer, Berlin, 1981, pp. 55–97. MR645907 (83c:03032)

[6] Oliver Deiser and Dieter Donder, *Canonical functions, non-regular ultrafilters and Ulam's problem on* ω_1, J. Symbolic Logic **68** (2003), no. 3, 713–739. MR2000073 (2004k:03095)

[7] Matthew Foreman, *An* $\aleph_1$*-dense ideal on* $\aleph_2$, Israel J. Math. **108** (1998), 253–290. MR1669368 (2000e:03140)

[8] M. Foreman, M. Magidor, and S. Shelah, *Martin's maximum, saturated ideals and nonregular ultrafilters. II*, Ann. of Math. (2) **127** (1988), no. 3, 521–545. MR942519 (90a:03077)

[9] Markus Huberich, *Non-regular ultrafilters*, Israel J. Math. **87** (1994), no. 1-3, 275–288. MR1286831 (95k:03085)

[10] A. Kanamori, *Weakly normal filters and irregular ultrafilters*, Trans. Amer. Math. Soc. **220** (1976), 393–399. MR0480041 (58 #240)

[11] Jussi Ketonen, *Nonregular ultrafilters and large cardinals*, Trans. Amer. Math. Soc. **224** (1976), 61–73. MR0419236 (54 #7260)

[12] Richard Laver, *Saturated ideals and nonregular ultrafilters*, Patras Logic Symposion (Patras, 1980), Stud. Logic Found. Math., vol. 109, North-Holland, Amsterdam, 1982, pp. 297–305. MR694265 (85e:03110)

[13] W. Hugh Woodin, *The axiom of determinacy, forcing axioms, and the nonstationary ideal*, de Gruyter Series in Logic and its Applications, vol. 1, Walter de Gruyter & Co., Berlin, 1999. MR1713438 (2001e:03001)

[14] Martin Zeman, *Inner models and large cardinals*, de Gruyter Series in Logic and its Applications, vol. 5, Walter de Gruyter & Co., Berlin, 2002. MR1876087 (2003a:03004)

E-mail address: `sean.cox@uni-muenster.de`

Institut für mathematische Logik und Grundlagenforschung, Universität Münster, Einsteinstrasse 62, 48149 Münster, Tel.: +49-251-83-33 790, Fax: +49-251-83-33 078