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Martin's Maximum, diagonal stationary set reflection, and well-determined generic embeddings

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1. Summary and Background

Summary

Strong forcing axioms like Martin's Maximum imply versions of the ***Diagonal Reflection Principle (DRP)****; this is a highly simultaneous form of stationary set reflection which yields interesting generic ultrapowers of the universe with critical point ω_2. Namely, these ultrapowers in some ways resemble generic ultrapowers by ideals whose associated posets are* proper. *Combined with absoluteness assumptions about partitions of $cof(\omega)$ into stationary sets, large portions of such generic embeddings must lie in the ground model.*

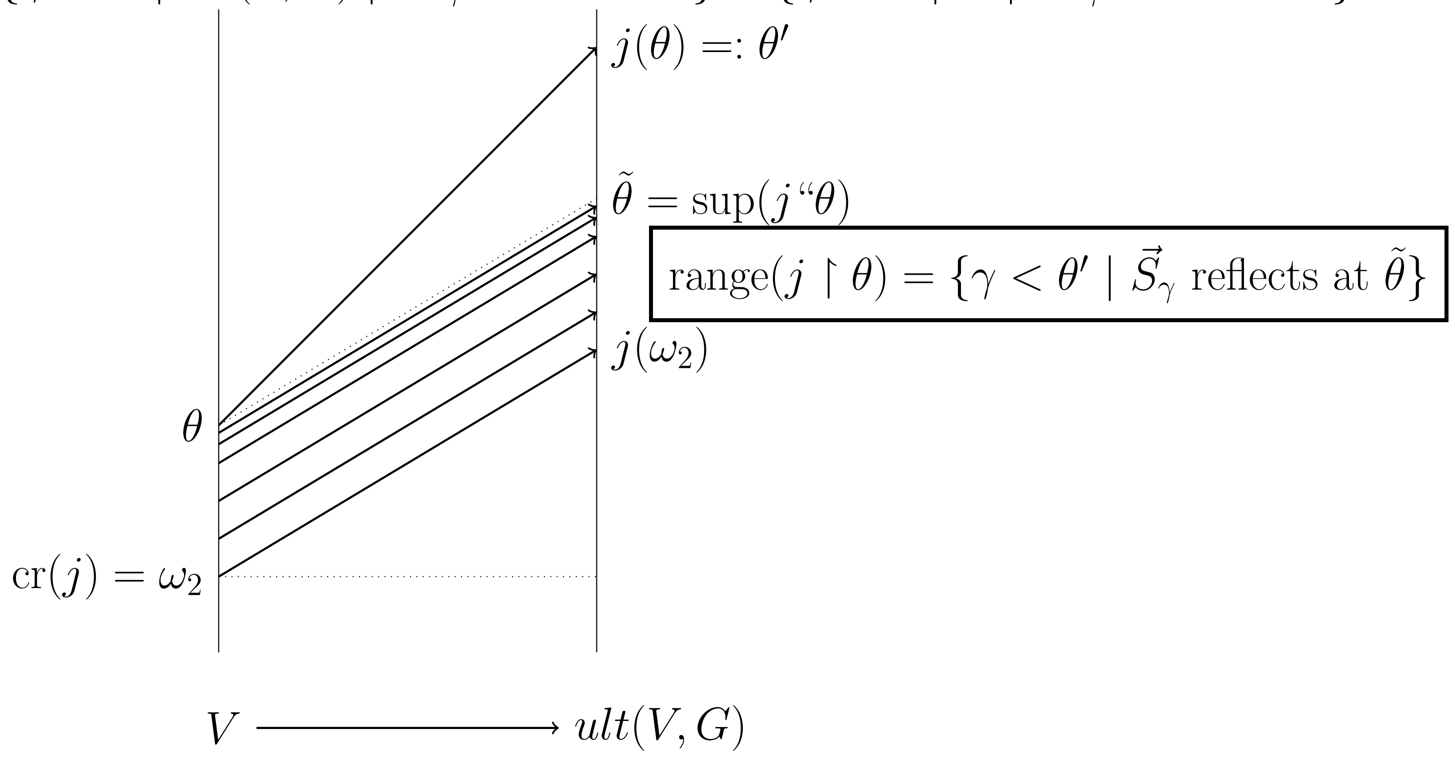
Suppose J is a normal ideal (e.g. the nonstationary ideal restricted to some stationary subset of $\wp(H_\theta)$ for some regular θ). Consider the partial order $\mathbb{P}_J := (J^+, \subset)$. If G is generic for $\mathbb{P}_J$, then G is a V-ultrafilter extending the dual of J; moreover G is normal with respect to sequences from V. The map $j_G : V \to_G ult(V,G)$ is called a *generic ultrapower by the ideal J*; here $ult(V,G)$ need not be wellfounded, but it is always wellfounded past the underlying set of J (e.g. if $J = NS \restriction \wp_{\omega_2}(H_\theta)$ then $ult(V,G)$ is wellfounded past θ). The critical point of j_G corresponds to the completeness of the ideal J in the ground model; so j_G can have small critical points like ω_1 or ω_2.

If the poset $\mathbb{P}_J = (J^+, \subset)$ has nice properties, then so will the generic embedding j_G. The relevant example for us is:

Example 1 *If $\mathbb{P}_J$ is a* proper poset *(i.e. forcing with it preserves stationary subsets of $[\lambda]^\omega$ for all uncountable λ), then $ult(V,G)$ is always wellfounded (see [5]) and V models various kinds of stationary set reflection.*

2. Properness of $\mathbb{P}_J$ and well-determined generic ultrapowers

One reason that Example 1 is interesting is that in the presence of reasonably absolute partitions of $cof(\omega)$ into stationary sets, large portions of generic ultrapower maps by such ideals are visible to the ground model. Suppose J measures subsets of $\wp_{\omega_2}(H_\theta)$, $\mathbb{P}_J$ is proper, and in $ult(V,G)$ there is a partition $\vec{S}$ of $\theta' := j_G(\theta) \cap cof(\omega)$ into θ'-many stationary sets, where $\vec{S}$ is also an element of the ground model (this is consistent with Example 1). Then $j_G \restriction \theta$ is an element of the ground model; it is the inverse of the collapsing map of $\{\gamma < \theta' \mid ult(V,G) \models \vec{S}_\gamma \text{ reflects at } \tilde{\theta}\} = \{\gamma < \theta' \mid V \models \vec{S}_\gamma \text{ reflects at } \tilde{\theta}\} \in V$:



In particular, there is a condition $S \in \mathbb{P}_J$ such that $j_G \restriction \theta$ does not depend on the choice of the generic $G \ni S$. Ideals with this property—i.e. that $j_G \restriction \theta$ does not depend on G—occur frequently in forcing extensions over models with large cardinals (see [3]).

3. The Diagonal Reflection Principle (DRP)

Example 1 is consistent with the Proper Forcing Axiom (PFA):

Theorem 2 *(C. [2]) It is consistent relative to a superhuge cardinal that PFA holds and there are lots of ideals J such that $\mathbb{P}_J$ is proper (Similarly for MM and semiproper posets of the form $\mathbb{P}_J$).*

However, the existence of J as in Example 1 does *not* follow from PFA (due to Beaudoin and Magidor independently). The following concept is closely related to Example 1:

Definition 3 *(C. [1])*
The ***Diagonal Reflection Principle (DRP)*** *states: for every regular $\theta \geq \omega_2$, there are stationarily many $M \in [H_{(\theta^\omega)^+}]^{\omega_1}$ such that every stationary element of M reflects to M. More precisely, $(\forall S \in M)(S \subset [\theta]^\omega$ is stationary $\to S \cap [M]^\omega$ is stationary$)$.*

If Γ is a class of structures, $DRP(\Gamma)$ means that in the statement of DRP we can also require that $M \in \Gamma$. Let IC_{ω_1} denote the class of **internally club** structures (M is internally club iff $M \cap [M]^\omega$ contains a club).

The following theorem says that DRP is a natural weakening of Example 1:

Theorem 4 *(C. [2]) $DRP(IC_{\omega_1})$ holds iff for every regular $\theta \geq \omega_2$ there is a generic ultrapower $j : V \to_G ult(V,G)$ such that $H_\theta^V \in (IC_{\omega_1})^{ult(V,G)}$ and all stationary subsets of $[\theta]^\omega$ remain stationary in $ult(V,G)$.*
(Though they need not remain stationary in $V[G]$!).

In particular, DRP can largely play the role of the properness of $\mathbb{P}_J$ in the diagram above.

4. Strong forcing axioms imply DRP; and some open problems

Variants of DRP follow from strong forcing axioms:

Theorem 5 *(C. [1])*

- *$FA^{+\omega_1}$(σ-closed posets) implies DRP.*
- *MM implies "weak" DRP; essentially this is DRP but only for stationary subsets of $\theta \cap cof(\omega)$, rather than $[\theta]^\omega$ (this strengthens results of Foreman [3] and of Larson [6]).*

Arguments from [4] show that MM implies RP_{ω_1}: that every ω_1-sized collection of stationary subsets of $[\theta]^\omega$ have a common reflecting set. RP_{ω_1} trivially follows from DRP. Moreover, Shelah proved that FA^{+1}(σ-closed posets) follows from MM. These results, along with Theorem 5, raise several questions:

Question 6 *Does MM imply DRP?*

Question 7 *Does MM imply $FA^{+\omega_1}$(σ-closed posets)? or even just FA^{+2}(σ-closed posets)?*

Question 8 *Does RP_{ω_1} imply DRP?*

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