Distinguished Lecture Series

Matthew Foreman on Large Cardinals, Set Theory, and Generic Elementary Embeddings

PROFESSOR MATTHEW FOREMAN (UNIVERSITY OF California, Irvine) gave the Distinguished Lecture Series from November 7-9, 2012 at the Fields Institute, as part of the *Thematic Program on Forcing and its Applications*.

The first lecture, titled *Large Cardinals: Who are they? What are they doing here? Why won't they go away?* dealt with the question of how mathematical knowledge is discovered, and the role of infinity in this discovery. Mathematical reasoning can usually be idealized as the application of first order logic to some recursive schema Σ of assumptions (axioms). The question of *what is a (formal) proof from* Σ is uncontroversial: a formal proof is a finite sequence of symbols obeying certain purely syntactic rules.* However, whether or not there is a single schema Σ of axioms that can serve as a sufficient background for all mathematical reasoning is far from settled, and Gödel's Incompleteness Theorem implies that any such (sufficiently useful) schema will necessarily be incomplete; that is, there will always be some statement which can neither be proved nor disproved from the axioms. Still there is natural and successful way to extend the usual axioms of mathematics with so-called *large cardinal axioms*. A large cardinal axiom asserts the existence of an infinite set which dominates smaller infinite sets, similar to the way the set of natural numbers dominates the finite sets. These axioms form a virtually linear hierarchy, and Foreman pointed out the "remarkable fact of nature" that, as far as we can tell, all natural axiom systems for set theory correspond (via the notion of *equiconsistency*) to some level of the large cardinal hierarchy. For example, there are large cardinal axioms that correspond exactly to the statement that every projective** set of real numbers is Lebesgue measurable.

The second lecture, titled *Does set theory have anything to do with mathematics?*, demonstrated that set theory influences even those areas of mathematics which deal with highly-structured objects. It is well known that many problems in general topology (e.g. Moore-Mrowka Problem) and abelian group theory (e.g. Whitehead Problem) are independent of the standard axiomatization of mathematics; however the objects studied in these areas have relatively little structure. Especially in recent years, set theoretic connections have been found with progressively more structured objects, such as Banach space theory, functional analysis, and compact smooth manifolds. Foreman discussed in detail the work of descriptive set theorists on the general problem of classifying equivalence relations via Borel reducibility and on the von Neumann program of classifying ergodic transformations. A sample result, due to Foreman and Weiss in 2010, is that for any $k \in [2,\infty]$ the isomorphism relation on the space of C^k measure preserving and ergodic diffeomorphisms on the 2-torus is complete analytic; this implies that even for concrete diffeomorphisms on the torus, classification is inherently impossible.

The third and final lecture, titled *Generic Elementary Embeddings*, dealt with large-cardinal-like behavior of small cardinals; here "small" refers to the size of sets typically encountered in other areas of mathematics, such as the first few infinite cardinals (i.e. $\aleph_n$ for each $n \in \mathbb{N}$) and their powersets. Such cardinals can never be full-fledged large cardinals; that is, they can never be the critical points of elementary embeddings into transitive models of set theory, **if** the embedding is definable in the set theoretic universe. But, for example, $\aleph_1$ *can* be the critical point of an elementary embedding *which is definable in a forcing extension* of the set-theoretic universe; the properties of such an embedding are typically related to the properties of the boolean algebra $\wp(\aleph_1)/\mathcal{J}$, where $\mathcal{J}$ is an *ideal*. Foreman concluded the last lecture of the Distinguished Lecture Series by outlining a striking result of his: Strong Chang Reflection, a combinatorial statement about the 4th uncountable cardinal, is roughly equiconsistent with a huge cardinal, a large cardinal concept far beyond the realm of other known equiconsistency phenomena.

Sean Cox (Virginia Commonwealth University)

* If Σ is recursive, there is an algorithm that can determine whether a given finite sequence of symbols is a legitimate proof from Σ. Though the vast majority of proofs in the literature are not formal proofs, they *could* straightforwardly be formalized, and in fact many complex proofs have been formalized (e.g. the Prime Number Theorem).

** A set $A \subseteq \mathbb{R}$ is called *projective* if A is obtained by a finite process of taking complements and projections, starting with some closed subset of $\mathbb{R}^n$ for some $n \in \mathbb{N}$.