Vopěnka's Principle and completeness of cotorsion pairs

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Outline

Cotorsion pairs (Salce)

Given a class $\mathcal{C}$ of modules:

$$^{\perp}\mathcal{C} := \{X \ : \ \mathrm{Ext}^1(X, C) = 0 \text{ for all } C \in \mathcal{C}\}$$

$$\mathcal{C}^{\perp} := \{Y \ : \ \mathrm{Ext}^1(C, Y) = 0 \text{ for all } C \in \mathcal{C}\}.$$

$(\mathcal{A}, \mathcal{B})$ is a **cotorsion pair** if $\mathcal{A} = {}^{\perp}\mathcal{B}$ and $\mathcal{A}^{\perp} = \mathcal{B}$. It is **complete** if for every module M there is a short exact sequence

$$0 \longrightarrow B \longrightarrow A \longrightarrow M \longrightarrow 0$$

with $A \in \mathcal{A}$ and $B \in \mathcal{B}$.

Salce's Problem (for **Ab**): Is every cotorsion pair in **Ab** complete?

Theorem (Eklof-Shelah [6])

Consistently, *no. You won't be able to prove (in* **ZFC** *alone) that Salce's Problem has an affirmative answer (assuming ZFC is consistent).*

Recent applications of **Vopěnka's Principle (VP)**

Theorem (C. [4])

If ZFC+VP is consistent, then it is consistent that whenever $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ is a cotorsion pair and ${}^{\perp}B$ is downward closed under pure submodules for all $B \in \mathcal{B}$, then $\mathfrak{C}$ is complete.

So you won't be able to solve Salce's Problem for **Ab**—or even for module categories over hereditary rings—in ZFC alone (unless VP is inconsistent).

The general version of Salce's Problem—*is it consistent that* **every** *cotorsion pair over* **every** *ring is complete?*—is still open.

A separate result, but still using "elementary submodel" arguments:

Theorem (C. [5])

VP implies that all classes of the form $\mathfrak{X}$-$\mathcal{GP}$ are **deconstructible** *(in particular,* **precovering***).*

Outline

History

Elementary submodel arguments are pervasive in forcing and infinite combinatorics.

I stumbled upon the statement of Kaplansky's Theorem about a year ago, and noticed that elementary submodel arguments gave a quick proof of it.

The next few slides are ZFC facts. Not all are relevant to Salce's Problem, but hopefully will give the flavor of these kinds of arguments.

(Almost) Elementary submodels of the universe

$\boldsymbol{(V, \in)}$: the universe of sets with its membership relation.

Fact (Löwnheim-Skolem; Montague)

Let λ be an infinite cardinal and Z any λ-sized set. Then there is an N such that:

1. $|N| = \lambda$;
2. $Z \subset N$; *and*
3. $\mathfrak{N} := \Big(N, \in \restriction (N \times N)\Big)$ *is "almost" an* **elementary submodel of the universe**. *We'll just write* $\mathfrak{N} \prec \boldsymbol{(V, \in)}$.

("almost")

Suppose $\mathfrak{N} \prec (V, \in)$. Then:

- $G \in \mathfrak{N}$ a group $\implies$ $\mathfrak{N} \cap G$ is a subgroup of G.
- $R \in \mathfrak{N}$ a ring $\implies$ $\mathfrak{N} \cap R$ is a subring of R.
- $M, R \in \mathfrak{N}$, $R \subset \mathfrak{N}$, and M is an R-module $\implies$ $\mathfrak{N} \cap M$ is an R-submodule of M.
- Suppose $M_\bullet = \left(M_k \to^{\pi_k} M_{k+1}\right)_{k \in \mathbb{Z}}$ is exact and $M_\bullet \in \mathfrak{N}$. In particular, $\boldsymbol{\pi_k} \in \boldsymbol{\mathfrak{N}}$ for each n. Define $\boldsymbol{M_\bullet} \restriction \boldsymbol{\mathfrak{N}}$ by:

$$\ldots \mathfrak{N} \cap M_{k-1} \xrightarrow{\pi_{k-1} \restriction \mathfrak{N}} \mathfrak{N} \cap M_k \xrightarrow{\pi_k \restriction \mathfrak{N}} \mathfrak{N} \cap M_{k+1} \ldots$$

  Then $M_\bullet \restriction \mathfrak{N}$:
  - makes sense (i.e., $\mathfrak{N}$ is closed under each π_k); and
  - is exact. (why?)

Suppose $\mathfrak{N} \prec (V, \in)$ (continued)

Suppose R is a ring, and (for simplicity) $R \cup \{R\} \subset \mathfrak{N}$.

1. If $F \in \mathfrak{N}$ is free, then $\mathfrak{N} \cap F$ and $\frac{F}{\mathfrak{N} \cap F}$ are both free.
   - *Idea*: Since $F \in \mathfrak{N} \prec (V, \in)$ there is free basis B with $B \in \mathfrak{N}$. Then show $\mathfrak{N} \cap F$ is "closed under B-supports".
2. Same holds when "free" replaced by "projective". See Lemma 3.3 of C. [5].
3. **Summary**: freeness and projectivity are "almost everywhere" closed under quotients and submodules.
4. Suppose $P_\bullet$ is exact complex of projective modules, and $P_\bullet \in \mathfrak{N}$. Recall $P_\bullet \restriction \mathfrak{N}$ denotes the exact complex

$$\dots \mathfrak{N} \cap P_{k-1} \longrightarrow \mathfrak{N} \cap P_k \longrightarrow \mathfrak{N} \cap P_{k+1} \dots$$

   By (2) above, each $\mathfrak{N} \cap P_k$ and $\frac{P_k}{\mathfrak{N} \cap P_k}$ are projective.

Induced SES of complexes when $P_\bullet \in \mathfrak{N} \prec (V, \in)$

$$
\begin{tikzcd}
\mathbf{0}_\bullet \arrow[d] & \dots \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \dots \\
\boldsymbol{P_\bullet \restriction \mathfrak{N}} \arrow[d] & \dots \arrow[r] & \mathfrak{N} \cap P_{-1} \arrow[r] \arrow[d] & \mathfrak{N} \cap P_0 \arrow[r] \arrow[d] & \mathfrak{N} \cap P_1 \arrow[r] \arrow[d] & \dots \\
\boldsymbol{P_\bullet} \arrow[d] & \dots \arrow[r] & P_{-1} \arrow[r] \arrow[d] & P_0 \arrow[r] \arrow[d] & P_1 \arrow[r] \arrow[d] & \dots \\
\boldsymbol{P_\bullet / \mathfrak{N}} \arrow[d] & \dots \arrow[r] & \frac{P_{-1}}{\mathfrak{N} \cap P_{-1}} \arrow[r] \arrow[d] & \frac{P_0}{\mathfrak{N} \cap P_0} \arrow[r] \arrow[d] & \frac{P_1}{\mathfrak{N} \cap P_1} \arrow[r] \arrow[d] & \dots \\
\mathbf{0}_\bullet & \dots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \dots
\end{tikzcd}
$$

Projectivity of $\frac{P_n}{\mathfrak{N} \cap P_n}$ ensures each column splits.
So for any module Q, $\mathrm{Hom}(-, Q)$ preserves exactness of each column.
So Hom$(-, Q)$ preserves exactness of the SES of complexes.

To summarize:

Lemma

Suppose:

1. $P_\bullet$ *is an exact complex of projective R-modules; and*
2. $R \cup \{R, P_\bullet\} \subset \mathfrak{N} \prec (V, \in)$.

Then for any module Q, applying $Hom(-, Q)$ *to the short exact sequence*

$$0_\bullet \longrightarrow P_\bullet \restriction \mathfrak{N} \longrightarrow P_\bullet \longrightarrow P_\bullet/\mathfrak{N} \longrightarrow 0_\bullet$$

yields another short exact sequence of complexes:

$$0_\bullet \longleftarrow Hom\big(P_\bullet \restriction \mathfrak{N}, Q\big) \longleftarrow Hom\big(P_\bullet, Q\big) \longleftarrow Hom\big(P_\bullet/\mathfrak{N}, Q\big) \longleftarrow 0_\bullet$$

The middle 3 complexes can fail to be exact; but if 2 out of 3 of them are exact, then so is the 3rd ("3-by-3 lemma").

Outline

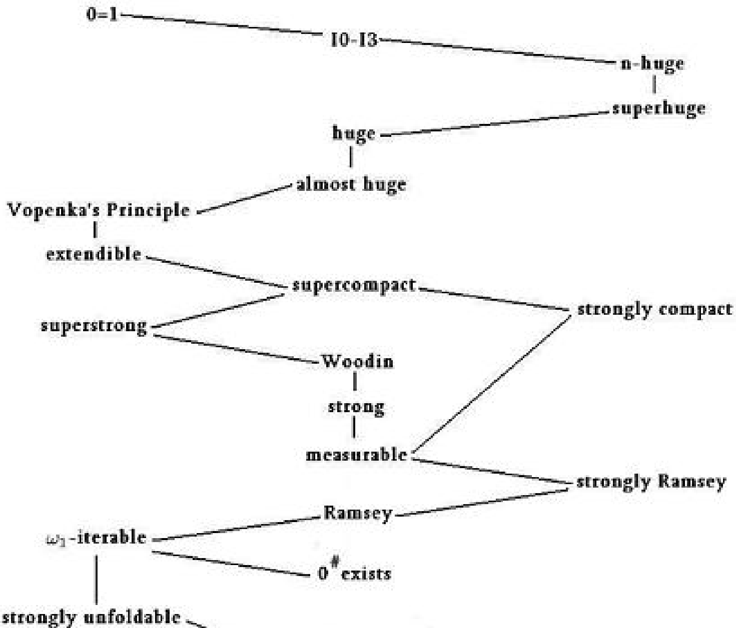
Vopěnka's Principle (**VP**): There is **NO** large, rigid class of graphs

From Adámek-Rosický [1]:

- VP: "a practical joke which misfired"
- Many characterizations of VP, e.g., **every subfunctor of an accessible functor is accessible**

VP sits in the *large cardinal hierarchy*.

from Kanamori's *Higher Infinite*



Consistency of positive solution to Salce's Problem

Theorem (C. [4])

If VP is consistent, then so is the scheme: "if $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ is a cotorsion pair, and ${}^{\perp}B$ is downward closed under pure submodules for all $B \in \mathcal{B}$, then $\mathfrak{C}$ is complete".

Corollary

It is consistent (relative to VP) that all cotorsion pairs over all hereditary rings are complete.

3 ingredients of the theorem

1. VP implies: if ${}^{\perp}B$ downward closed under pure submodules for all $B \in \mathcal{B}$, then there is a **set** $\mathcal{B}_0 \subset \mathcal{B}$ such that

$$ {}^{\perp}\mathcal{B} = {}^{\perp}\mathcal{B}_0. $$

   Put another way: If $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ is a cotorsion pair and ${}^{\perp}B$ is downward closed under pure submodules for all $B \in \mathcal{B}$, then $\mathfrak{C}$ is **cogenerated by a set** (acc. to Göbel-Trlifaj; "generated by a set" acc. to Salce and Eklof-Trlifaj).

2. CON(VP) implies consistency of VP plus

$$ \forall \lambda \in \mathrm{REG} \cap [\aleph_1, \infty) \ \ \forall S \subseteq \lambda \ \ S \text{ stationary } \implies \Diamond_\lambda(S) \ \ (*) $$

   (uses "VP-preservation" forcing theorem of Brooke-Taylor [3])

3. (Šaroch-Trlifaj [10]; Eklof-Trlifaj [7],[8]): if (*) holds, then all cotorsion pairs that are cogenerated by a set, and whose left coordinate is downward closed under pure submodules, are also generated by a set (hence, complete).

Sketch of 1st part

Lemma (special case of Lemma 4.1 of C. [4])

Assume VP. Suppose R is a ring and P is an ℓ-ary class relation on R-modules. Then there is a cardinal κ_P such that $|R| < \kappa_P$ and, for any ℓ-tuple of R-modules $M_1, \ldots, M_\ell$: there is an $\mathfrak{N}$ such that

$$R \cup \{R, M_1, \ldots, M_\ell\} \subset \mathfrak{N},$$

$$\mathfrak{N} \prec (V, \in),$$

$$|\mathfrak{N}| < \kappa_P,$$

and

$$(M_1, \ldots, M_\ell) \in P \iff \left(\mathfrak{N} \cap M_1, \ldots, \mathfrak{N} \cap M_\ell\right) \in P.$$

(uses Bagaria's [2] characterization of VP in terms of C(n)-extendible cardinals)

Assume VP, and ${}^\perp B$ downward closed under pure submodules for all $B \in \mathcal{B}$

Let $P := \Big\{(M,B) \ : \ B \in \mathcal{B} \text{ and } M \notin {}^\perp B\Big\}$, and $\kappa := \kappa_P$ be as in the lemma. **CLAIM: ${}^\perp \boldsymbol{\mathcal{B}} = {}^\perp(\boldsymbol{\mathcal{B}}^{<\kappa})$**. The $\subseteq$ direction is trivial. To see $\supseteq$, suppose $M \notin {}^\perp B$ where $B \in \mathcal{B}$; so $\boldsymbol{(M,B) \in P}$.

1. By the lemma, there is $\mathfrak{N}$ with $|\mathfrak{N}| < \kappa$, $R \cup \{R, M, B\} \subset \mathfrak{N} \prec (V, \in)$, and $\big(\mathfrak{N} \cap \boldsymbol{M}, \mathfrak{N} \cap \boldsymbol{B}\big) \in \boldsymbol{P}$. I.e.,
$$\mathfrak{N} \cap B \in \mathcal{B} \text{ and } \mathfrak{N} \cap M \notin {}^\perp\big(\mathfrak{N} \cap B\big).$$
2. In particular, $\mathfrak{N} \cap B \in \mathcal{B}^{<\kappa}$, and ${}^\perp(\mathfrak{N} \cap B)$ is downward closed under pure submodules.
3. $R \cup \{R, M\} \subset \mathfrak{N} \prec (V, \in) \implies \mathfrak{N} \cap M$ is a pure submodule of M (routine). By downward closure of ${}^\perp(\mathfrak{N} \cap B)$, and since $\mathfrak{N} \cap M \notin {}^\perp(\mathfrak{N} \cap B)$, it follows that $M \notin {}^\perp(\mathfrak{N} \cap B)$. So $M \notin {}^\perp(\mathcal{B}^{<\kappa})$.

Outline

Deconstructible and precovering classes

Definition

A class $\mathcal{G}$ of modules is **deconstructible** if there exists a cardinal κ such that every $G \in \mathcal{G}$ can be written as the union of a smooth, well-ordered chain $\vec{G}$, such that G_0, and each $G_{\alpha+1}/G_\alpha$, are $< \boldsymbol{\kappa}$-presented members of $\mathcal{G}$.

Theorem (Eklof-Trlifaj [8])

If $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$ is a cotorsion pair and $\mathcal{A}$ is deconstructible, then $\mathfrak{C}$ is complete.

Theorem (Saorín-Šťovíček [9])

A deconstructible class that is closed under transfinite extensions is special precovering.

A new "top-down" characterization of deconstructibility

Theorem (C. [5])

Consider the following statements about a class $\mathcal{G}$:

1. *$\mathcal{G}$ is deconstructible*
2. *There is a κ such that whenever $\mathfrak{N} \prec (V, \in)$ and* **$\mathfrak{N} \cap \kappa$ is transitive**, *then*

$$G \in \mathfrak{N} \cap \mathcal{G} \implies \mathfrak{N} \cap G \in \mathcal{G} \text{ and } \frac{G}{\mathfrak{N} \cap G} \in \mathcal{G}.$$

The (1) $\Longleftarrow$ (2) direction always holds.
If $\mathcal{G}$ is closed under transfinite extensions, then (1) $\iff$ (2).

Examples of deconstructible classes

"whenever $\mathfrak{N} \prec (V, \in)$ and $\mathfrak{N} \cap \kappa$ is transitive, then

$$G \in \mathfrak{N} \cap \mathcal{G} \implies \mathfrak{N} \cap G \in \mathcal{G} \text{ and } \frac{G}{\mathfrak{N} \cap G} \in \mathcal{G}."$$

Examples:

1. Kaplansky's Theorem: $\mathcal{G} = \{\text{projectives}\}$, $\kappa = \aleph_1$.
2. Enochs' Theorem: $\mathcal{G} = \{\text{flats}\}$, $\kappa = |R|^+$.
3. (C. [5]): $\mathcal{G} = \boldsymbol{\mathfrak{X}\text{-}\mathcal{GP}}$, $\kappa =$ a cardinal given by Vopěnka's Principle that depends on $\mathfrak{X}$. Special cases:
   - Gorenstein Projectives ($\boldsymbol{\mathcal{GP}}$) when $\mathfrak{X} = \{\text{projectives}\}$
   - Ding Projectives ($\boldsymbol{\mathcal{DP}}$) when $\mathfrak{X} = \{\text{flats}\}$

   The $\mathcal{GP}$ result was proved earlier by Šaroch (using "only" strongly compact cardinals).

Outline

Open Problems

Cotorsion pairs:

Problem (General Salce's Problem)

Is there a (ZFC-provable) example of a ring R and a non-complete cotorsion pair in R-Mod?

Deconstructibility/precovering:

Problem

Do either "$\mathcal{GP}$ is deconstructible" or "$\mathfrak{X}$-$\mathcal{GP}$ is deconstructible for all $\mathfrak{X}$" carry large cardinal consistency strength?

Problem

To which categories does the characterization of deconstructibility generalize? (My paper only did it up through complexes of modules)

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[4] Sean Cox, *Salce's problem on cotorsion pairs is undecidable*, submitted, available at `https://arxiv.org/abs/2103.06687`.

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[7] Paul C. Eklof and Jan Trlifaj, *Covers induced by Ext*, J. Algebra **231** (2000), no. 2, 640–651.

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