Gorenstein Homological algebra and elementary submodels

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Outline

Classical homological algebra and projective resolutions

Given a module M, a **projective resolution of M** is an **exact** sequence of homomorphisms

$$\ldots \longrightarrow P_2 \xrightarrow{\pi_2} P_1 \xrightarrow{\pi_1} P_0 \xrightarrow{\pi_0} M \longrightarrow 0$$

where each P_n is a **projective** module.

Used to define/compute certain invariants/functors, e.g. to compute $\mathbf{Ext}(M, -)$.

There are *many* projective resolutions of M, but the computation of $\operatorname{Ext}(M, -)$ does **NOT** depend on the choice of the projective resolution!

Homological algebra "relative to the class $\mathcal{G}$" ...

...attempts to do homological algebra, but where some class $\mathcal{G}$ is used *instead of* the class of projective modules.

E.g., one considers **$\mathcal{G}$-resolutions of M**:

$$\dots \longrightarrow G_2 \xrightarrow{\pi_2} G_1 \xrightarrow{\pi_1} G_0 \xrightarrow{\pi_0} M \longrightarrow 0$$

where each $G_n \in \mathcal{G}$.

However, in order for these to be useful, one needs that $\mathcal{G}$ is a **precovering** ("right-approximating") class.

Precovering classes

(Auslander-Reiten-Smalø; Enochs-Xu)

Suppose R is a ring and $\mathcal{G}$ a class of R-modules. Let M be any R-module. A **$\mathcal{G}$-precover of M** is a homomorphism

$$G \xrightarrow{\pi} M$$

such that:

1. $G \in \mathcal{G}$; and
2. $\forall G' \in \mathcal{G} \ \ \forall \pi' \in \mathrm{HOM}(G', M)$: π' factors through π

$$\begin{tikzcd} G \arrow[r, "\pi"] & M \\ G' \arrow[u, dotted] \arrow[ur, "\pi'"'] & \end{tikzcd}$$

$\mathcal{G}$ is a **precovering class** if every module has a $\mathcal{G}$-precover.

Case of interest: $\mathcal{G}$ = the "Gorenstein Projectives"

Question ([2], [3], [4], [5], [6], [7], [8], [9], [10], [11], [12], [13], [14])

Is the class of Gorenstein Projective modules a precovering class (over every ring)?

Gillespie [8]:

> *"...the most fundamental question in Gorenstein homological algebra is whether or not Gorenstein injective preenvelopes (***resp. Gorenstein projective precovers***) exist."*

Theorem (Šaroch)

It is consistent (relative to large cardinals) that Gorenstein Projectives always form a precovering class.

I later proved it (independently). Our proofs generalize in different directions.

Outline

An illustrative example of the use of elementary submodels

An R-module F is **free** if there is an R-linearly independent $B \subseteq F$ that generates F.

Free modules are not, in general, closed under submodules or under quotients!

- **Example of failure under quotients:** $R = \mathbb{Z}$. Both $\mathbb{Z}$ and $2\mathbb{Z}$ are free $\mathbb{Z}$-modules, but $\mathbb{Z}/2\mathbb{Z}$ is not a free $\mathbb{Z}$-module.
- **Example of failure under submodules:** Let $R = \mathbb{Z}/4\mathbb{Z}$, and set $F := R$. Then F is a free $\mathbb{Z}/4\mathbb{Z}$-module, but its submodule $2\mathbb{Z}/4\mathbb{Z}$ is **not** a free $\mathbb{Z}/4\mathbb{Z}$-module.

An illustrative example (cont.)

Lemma (“almost everywhere” closure of freeness under submodules and quotients)

If F is a free R-module and $\{R, F\} \subset \mathfrak{N} \prec_1 (V, \in)$, then both $\langle \mathfrak{N} \cap F \rangle$ and $\frac{F}{\langle \mathfrak{N} \cap F \rangle}$ are free.

The use of $\mathfrak{N} \prec_1 (V, \in)$ will end up being vast overkill, but such $\mathfrak{N}$'s will be convenient later on.

Club-many submodules of F are of that form anyway

An illustrative example (cont.)

Fix free basis B of F; **WLOG** $\boldsymbol{B} \in \mathfrak{N}$ b/c $\{F, R\} \subset \mathfrak{N} \prec_1 (V, \in)$.

$$x \in F \quad \Longrightarrow \quad x = \sum_{b \in \mathbf{sprt}_{\boldsymbol{B}}(\boldsymbol{x})} r_b^x b.$$

DEF: $X \subset F$ is "closed under B-supports" if $\mathrm{sprt}_B(x) \subset X$ for every $x \in X$.

An illustrative example (cont.)

CLAIM 1: X closed under B-supports $\implies$ both X and F/X are free.

Proof: obviously this guarantees that $B \cap X$ freely generates X. We claim that F/X is freely generated by $\{b + X \ : \ b \in B \setminus X\}$, and the only nontrivial thing to check is the freeness:

- Say $\sum_{i=1}^{k} r_i(b_i + X) = 0 + X$, each $b_i \in B \setminus X$ (distinct)
- So $\exists x \in X, \ \ x = \sum_{i=1}^{k} r_i b_i$.
- Then
$$\mathrm{sprt}_B(x) \subseteq \{b_1, \ldots, b_k\}.$$
- By closure of X under B-supports, and since each $b_i \notin X$, it follows that $\mathrm{sprt}_B(x)$ is empty, so $\boldsymbol{x = 0}$.
- By linear independence of B, it follows that each $r_i = 0$.

An illustrative example (cont.)

CLAIM 2: $\langle \mathfrak{N} \cap F \rangle$ is closed under B-supports.

- Consider any $z = \sum_{i=1}^{k} r_i x_i$ where $r_i \in R$ and $x_i \in \mathfrak{N} \cap F$.
- Then $\mathrm{sprt}_B(z) \subseteq \bigcup_{i=1}^{k} \mathrm{sprt}_B(x_i)$.
- Since $\{B, x_1, \ldots, x_k\} \subset \mathfrak{N} \prec_1 (V, \in)$, it follows that $\mathrm{sprt}_B(x_i) \in \mathfrak{N}$ for each $i = 1, \ldots, k$.
- Hence,
$$\bigcup_{i=1}^{k} \mathrm{sprt}_B(x_i) \in \mathfrak{N}.$$
- That union is a finite set, so it is also a subset of $\mathfrak{N}$. So $\mathrm{sprt}_B(z) \subset \mathfrak{N}$.

A similar proof (using "dual basis" of proj. modules)

Lemma

If P is a projective R-module and $\{R, P\} \subset \mathfrak{N} \prec_1 (V, \in)$, then both $\langle \mathfrak{N} \cap P \rangle$ and $\frac{P}{\langle \mathfrak{N} \cap P \rangle}$ are projective.

Even better: suppose $P_\bullet = \left(P_n \to^{\pi_n} P_{n+1} \right)_{n \in \mathbb{Z}}$ is a (long) exact sequence of projective R-modules.
Suppose $\{R, P_\bullet\} \subset \mathfrak{N} \prec_1 (V, \in)$. Then $\mathfrak{N}$ is closed under each π_n, so it makes sense to form (for simplicity assume $R \subset \mathfrak{N}$ too):

$$\boldsymbol{P_\bullet} \restriction \boldsymbol{\mathfrak{N}} : \ldots \longrightarrow \mathfrak{N} \cap P_n \xrightarrow{\pi_n \restriction \mathfrak{N}} \mathfrak{N} \cap P_{n+1} \longrightarrow \ldots$$

This is still exact b/c $\mathfrak{N} \prec_1 (V, \in)$; and by lemma above, **both** $\mathfrak{N} \cap \boldsymbol{P_n}$ **and** $\frac{\boldsymbol{P_n}}{\mathfrak{N} \cap \boldsymbol{P_n}}$ **are projective**.

SES of complexes induced by $\mathfrak{N}$

$$\begin{array}{ccccccccc}
\mathbf{0}_\bullet & & \cdots \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow \cdots \\
\downarrow & & & \downarrow & & \downarrow & & \downarrow & \\
\boldsymbol{P}_\bullet \restriction \mathfrak{N} & & \cdots \longrightarrow & \mathfrak{N} \cap P_{-1} & \longrightarrow & \mathfrak{N} \cap P_0 & \longrightarrow & \mathfrak{N} \cap P_1 & \longrightarrow \cdots \\
\downarrow & & & \downarrow & & \downarrow & & \downarrow & \\
\boldsymbol{P}_\bullet & & \cdots \longrightarrow & P_{-1} & \longrightarrow & P_0 & \longrightarrow & P_1 & \longrightarrow \cdots \\
\downarrow & & & \downarrow & & \downarrow & & \downarrow & \\
\boldsymbol{P}_\bullet/\mathfrak{N} & & \cdots \longrightarrow & \frac{P_{-1}}{\mathfrak{N} \cap P_{-1}} & \longrightarrow & \frac{P_0}{\mathfrak{N} \cap P_0} & \longrightarrow & \frac{P_1}{\mathfrak{N} \cap P_1} & \longrightarrow \cdots \\
\downarrow & & & \downarrow & & \downarrow & & \downarrow & \\
\mathbf{0}_\bullet & & \cdots \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow \cdots
\end{array}$$

Projectivity of $\frac{P_n}{\mathfrak{N} \cap P_n}$ ensures $\mathfrak{N} \cap P_n$ is **direct summand** of P_n.
So for any Q, every $\pi : \mathfrak{N} \cap P_n \to Q$ lifts to some $\widetilde{\pi} : P_n \to Q$.
("Hom$(-, Q)$ preserves exactness of each column.")
So Hom$(-, Q)$ preserves exactness of the SES of complexes.

To summarize:

Lemma

Suppose:

1. $P_\bullet$ *is an exact complex of projective* R*-modules; and*
2. $R \cup \{R, P_\bullet\} \subset \mathfrak{N} \prec_1 (V, \in)$.

Then for any module Q*, applying* $Hom(-, Q)$ *to the short exact sequence*

$$0_\bullet \longrightarrow P_\bullet \restriction \mathfrak{N} \longrightarrow P_\bullet \longrightarrow P_\bullet/\mathfrak{N} \longrightarrow 0_\bullet$$

yields another short exact sequence of complexes:

$$0_\bullet \longleftarrow Hom\big(P_\bullet \restriction \mathfrak{N}, Q\big) \longleftarrow Hom\big(P_\bullet, Q\big) \longleftarrow Hom\big(P_\bullet/\mathfrak{N}, Q\big) \longleftarrow 0_\bullet$$

The individual terms (complexes) in the latter can fail to be exact. But if 2 out of 3 of them are exact, then so is the 3rd ("3-by-3 lemma")

Outline

Deconstructibility: a sufficient condition for precovering

Theorem (Kaplansky)

Every projective module is a direct sum of countably-generated projective modules.

"class of projectives is $\aleph_0$-decomposable"

- Decomposability is rare.
- Enochs, Eklof and others isolated a weaker notion: **deconstructibility**.
  - a class $\mathcal{G}$ is **deconstructible** if there exists a cardinal κ such that every $G \in \mathcal{G}$ can be written as the union of a smooth, well-ordered chain $\vec{G}$, such that G_0, and each $G_{\alpha+1}/G_\alpha$, are $<$ $\boldsymbol{\kappa}$**-generated** members of $\mathcal{G}$.
- (Saorin-Stovicek): Deconstructible* classes are precovering.
  - Closely related to Quillen's *Small Object Argument*

A new "top-down" characterization of deconstructibility

Theorem (C. [1])

The following are equivalent (roughly):

1. $\mathcal{G}$ *is a deconstructible class of* R*-modules.*
2. *There exists a regular uncountable cardinal* κ *such that: whenever* $R \in \mathfrak{N} \prec_1 (V, \in)$ *and* $\mathfrak{N} \cap \kappa$ *is transitive, then*

$$G \in \mathfrak{N} \cap \mathcal{G} \implies \mathfrak{N} \cap G \in \mathcal{G} \text{ and } \frac{G}{\mathfrak{N} \cap G} \in \mathcal{G}.$$

$\Uparrow$ provides "top-down" way of verifying deconstructibility, and hence the precovering property.

Part 2 could also be expressed using (variant of) **Stationary Logic**

Outline

Gorenstein Projectives (Enochs-Xu; Aus.-Reiten-Smalø)

A **complete projective resolution** is a complex $P_\bullet$ of modules

$$\dots \longrightarrow P_{-1} \longrightarrow P_0 \longrightarrow P_1 \longrightarrow \dots$$

such that:

- Each P_n is projective; and
- $P_\bullet$ is exact, and also "$\mathrm{Hom}(-, \mathrm{Proj})$-exact"; i.e., for every projective module Q, the complex $\mathrm{HOM}(P_\bullet, Q)$:

$$\dots \mathrm{Hom}(P_{-1}, Q) \longleftarrow \mathrm{Hom}(P_0, Q) \longleftarrow \mathrm{Hom}(P_1, Q) \dots$$

  is still exact.

A module G is **Gorenstein Projective** if $G = \ker(P_0 \to P_1)$ for some complete projective resolution $P_\bullet$.

The Main Theorem

Theorem (C. [1])

Let $\mathcal{GP}_R$ denote the class of Gorenstein Projective R-modules. Suppose $\kappa > |R|$ is supercompact. Then whenever $R \in \mathfrak{N} \prec_1 (V, \in)$ and $\mathfrak{N} \cap \kappa$ is transitive, the following implication holds:

$$G \in \mathfrak{N} \cap \mathcal{GP}_R \implies \mathfrak{N} \cap G \in \mathcal{GP}_R \text{ and } \frac{G}{\mathfrak{N} \cap G} \in \mathcal{GP}_R.$$

(Then by the new characterization of deconstructibility a few slides ago, $\mathcal{GP}_R$ is deconstructible, and hence precovering.)

- **NTS**: If $P_\bullet \in \mathfrak{N}$ is a complete projective resolution, then so are $P_\bullet \restriction \mathfrak{N}$ and $P_\bullet / \mathfrak{N}$.
- By nice quotient behavior of comp. proj. resolutions w.r.t. elementary submodels of $(V, \in)$, suffices to show this for **stationarily many** $\mathfrak{N} \in \wp_\kappa(H_\theta)$.

Outline of proof

1. Supercompactness of κ yields stationarily many $\mathfrak{N} \in \wp_\kappa(H_\theta)$ whose Mostowski collapse $\overline{\mathfrak{N}}$ is hereditarily closed (i.e., of the form H_μ).
2. Consider such an $\mathfrak{N}$, and any c.p.r. $P_\bullet \in \mathfrak{N}$.
   - NTS that $P_\bullet \restriction \mathfrak{N}$ and $P_\bullet / \mathfrak{N}$ are both $\mathrm{Hom}(-, \mathrm{Proj})$-exact.
   - By earlier part of talk, suffices to just show it for $P_\bullet \restriction \mathfrak{N}$.
3. $\overline{P_\bullet}$:= image of $P_\bullet$ under trans. collapse; note $\overline{P_\bullet} \simeq P_\bullet \restriction \mathfrak{N}$.
4. $\overline{\mathfrak{N}} \models$ "$\overline{P_\bullet}$ is $\mathrm{HOM}(-, \mathrm{Proj})$-exact" by elementarity of Mostowski collapsing map
   - Since $\overline{\mathfrak{N}}$ is hereditarily closed, quoted statement is upward absolute to V (ZFC fact; another elementary submodel argument)
   - So $\overline{P_\bullet} \simeq P_\bullet \restriction \mathfrak{N}$ is $\mathrm{HOM}(-, \mathrm{Proj})$-exact.

Technical remark

Supercompactness of κ gave us "positively many" $\mathcal{N} \prec_1 (V, \in)$ with those desired properties.

To extrapolate to "almost all" $\mathfrak{N}$ (with trans. intersection with κ), make use of nice quotient and extension behavior of complete projective resolutions w.r.t. elementary submodels.

Maximum Deconstructibility

Definition (C.)

A class $\mathcal{G}$ of complexes of modules is **eventually almost everywhere closed under quotients** if for all large regular κ: whenever $\mathfrak{N} \prec_1 (V, \in)$ and $\mathfrak{N} \cap \kappa$ is transitive, then for all $M_\bullet \in \mathfrak{N}$:

$$\{M_\bullet, M_\bullet \restriction \mathfrak{N}\} \subset \mathcal{G} \implies M_\bullet / \mathfrak{N} \in \mathcal{G}.$$

Example: complete projective resolutions (and many more).

Corollary of my new characterization of deconstructibility: eventual a.e. closure under quotients is a *necessary condition* for deconstructibility*

Theorem (C.)

Assume (approximately) Vopenka's Principle. Then this necessary condition is also sufficient (for classes closed under transf. extensions).
In particular, all classes of the form "$\mathfrak{X}$-$\mathcal{GP}$" are deconstructible.

Open Problems

So if large cardinals exist, then $\mathcal{GP}$ is precovering over every ring. Can you prove it in ZFC? Does it have large cardinal strength? What if the ring is countable?

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