Homological algebra, elementary submodels, and stationary logic

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Overview

Two recent solutions of problems from the algebra literature:

Theorem (C. [4])

"Salce's Problem" about completeness of cotorsion pairs is independent of ZFC (assuming consistency of Vopěnka's Principle)

Theorem (C. [3])

Vopěnka's Principle implies affirmative solution to many precovering problems in Gorenstein Homological algebra

Much of the analysis is in ZFC alone.
Key tool: set-theoretic *elementary submodel* arguments.

Outline

Illustrative example of the use of elementary submodels

An R-module F is **free** if there is an R-linearly independent $B \subseteq F$ that generates F.

Free modules are not, in general, closed under submodules or under quotients!

- **Example of failure under quotients:** $R = \mathbb{Z}$. Both $\mathbb{Z}$ and $2\mathbb{Z}$ are free (as $\mathbb{Z}$-modules), but $\mathbb{Z}/2\mathbb{Z}$ is *not* free.
- **Example of failure under submodules:** Let $R = \mathbb{Z}/4\mathbb{Z}$, and set $F := R$. Then F is a free R-module, but its submodule $2\mathbb{Z}/4\mathbb{Z}$ is **not** free, because

$$\underbrace{(2+4\mathbb{Z})}_{\text{a nonzero scalar from } R} \quad \underbrace{(2+4\mathbb{Z})}_{\substack{\text{The only nonzero element of}\\ \text{the module } 2\mathbb{Z}/4\mathbb{Z}}} \quad = 0$$

An illustrative example (cont.)

But on a positive note:

Lemma ("almost everywhere" closure of freeness under submodules and quotients)

If F is a free R-module and $R, F \in \mathfrak{N} \prec_1 (V, \in)$, then both $\langle \mathfrak{N} \cap F \rangle$ and $\frac{F}{\langle \mathfrak{N} \cap F \rangle}$ are free.

$\mathfrak{N} \prec_1 (V, \in)$ is vast overkill here, but such $\mathfrak{N}$'s will be convenient later on.

An illustrative example (cont.)

Proof of lemma: fix free basis B of F; **WLOG $\boldsymbol{B} \in \mathfrak{N}$** b/c $F, R \in \mathfrak{N} \prec_1 (V, \in)$.

$$x \in F \quad \implies \quad x = \sum_{b \in \mathbf{sprt}_{\boldsymbol{B}}(\boldsymbol{x})} r_b^x b.$$

DEF: $X \subset F$ is "closed under B-supports" if $\text{sprt}_B(x) \subset X$ for every $x \in X$. **Basic algebra exercise:** if X is closed under B-supports then both X and F/X are free.

CLAIM: $\langle \mathfrak{N} \cap F \rangle$ is closed under B-supports.

An illustrative example (cont.)

CLAIM: $\langle \mathfrak{N} \cap \boldsymbol{F} \rangle$ is closed under $\boldsymbol{B}$-supports. Let $z \in \langle \mathfrak{N} \cap F \rangle$.

- Then $z = \sum_{i=1}^{k} r_i x_i$ for some $r_i \in R$ and $x_i \in \mathfrak{N} \cap F$.
- Then

$$\operatorname{sprt}_B(z) \subseteq \bigcup_{i=1}^{k} \operatorname{sprt}_B(x_i) \tag{1}$$

- Each $\operatorname{sprt}_B(x_i) \in \mathfrak{N}$ $\big($ because $B, x_i \in \mathfrak{N} \prec_1 (V, \in)$ $\big)$
  - Hence, each $\operatorname{sprt}_B(x_i) \subset \mathfrak{N}$ because finite, even countable, elements of $\mathfrak{N}$ are always subsets of $\mathfrak{N}$.
  - Together with (1) this yields

$$\operatorname{sprt}_B(z) \subset \mathfrak{N}.$$

Where elementary submodels *really* help

Often encounter (long) exact sequences of projective modules:

$$P_\bullet = \left(P_n \xrightarrow{\pi_n} P_{n+1} \right)_{n \in \mathbb{Z}}$$

If $P_\bullet \in \mathfrak{N} \prec_1 (V, \in)$, define

$$\boldsymbol{P_\bullet} \restriction \boldsymbol{\mathfrak{N}} := \left(\mathfrak{N} \cap P_n \xrightarrow{\pi_n \restriction \mathfrak{N}} \mathfrak{N} \cap P_{n+1} \right)_{n \in \mathbb{Z}}$$

(still exact, by elementarity of $\mathfrak{N}$). Induces a short exact sequence of complexes of projective modules:

$$0_\bullet \longrightarrow P_\bullet \restriction \mathfrak{N} \longrightarrow P_\bullet \longrightarrow P_\bullet / \mathfrak{N} \longrightarrow 0_\bullet \qquad (+)$$

Elementarity of $\mathfrak{N}$ ensures that $(+)$ behaves well w.r.t. certain functors; e.g., its exactness is preserved by the contravariant functor $\mathrm{Hom}(-, M)$ (for any M).

Short exact sequence of complexes induced by $\mathfrak{N}$ and $P_\bullet$

$$
\begin{tikzcd}
\mathbf{0}_\bullet \arrow[d] & \dots \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \dots \\
\boldsymbol{P_\bullet} \restriction \mathfrak{N} \arrow[d] & \dots \arrow[r] & \mathfrak{N} \cap P_{-1} \arrow[r] \arrow[d] & \mathfrak{N} \cap P_0 \arrow[r] \arrow[d] & \mathfrak{N} \cap P_1 \arrow[r] \arrow[d] & \dots \\
\boldsymbol{P_\bullet} \arrow[d] & \dots \arrow[r] & P_{-1} \arrow[r] \arrow[d] & P_0 \arrow[r] \arrow[d] & P_1 \arrow[r] \arrow[d] & \dots \\
\boldsymbol{P_\bullet}/\mathfrak{N} \arrow[d] & \dots \arrow[r] & \frac{P_{-1}}{\mathfrak{N}\cap P_{-1}} \arrow[r] \arrow[d] & \frac{P_0}{\mathfrak{N}\cap P_0} \arrow[r] \arrow[d] & \frac{P_1}{\mathfrak{N}\cap P_1} \arrow[r] \arrow[d] & \dots \\
\mathbf{0}_\bullet & \dots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \dots
\end{tikzcd}
$$

Projectivity of $\frac{P_n}{\mathfrak{N}\cap P_n}$ ensures $\mathfrak{N} \cap P_n$ is **direct summand** of P_n.
So for any M, every $\pi : \mathfrak{N} \cap P_n \to M$ lifts to some $\widetilde{\pi} : P_n \to M$.
("Hom$(-, M)$ preserves exactness of each column.")
So Hom$(-, M)$ preserves exactness of the SES of complexes.

Outline

The concept of a **deconstructible class of modules** . . .

. . . was isolated around 2000 (Eklof-Trlifaj [5]; Bican-El Bashir-Enochs [2]).

My 2 theorems at the beginning both go by showing that a certain class of modules is deconstructible.

Decomposability versus Deconstructibility

Definition (Eklof, Enochs, Trlifaj)

A class $\mathcal{C}$ of modules is **κ-deconstructible** if for every $M \in \mathcal{C}$, M can be written as the union of some $\subseteq$-increasing and $\subseteq$-continuous chain $\vec{M} = \langle M_\xi \ : \ \xi < \zeta \rangle$ (for some ζ depending on M) such that M_0, and each $M_{\xi+1}/M_\xi$, is:

1. in $\mathcal{C}$; and
2. $< \kappa$-generated.

$\mathcal{C}$ is **deconstructible** if it is κ-deconstructible for some κ.

κ-deconstructibility is considerably weaker than the rare property of κ-de**compos**ability; the latter requires each M_ξ to also be a direct summand of $M_{\xi+1}$.
Kaplansky famously proved in 1950s that the class of projective modules is $\aleph_1$-decomposable.

A new "top-down" characterization of deconstructibility

Theorem (C. [3])

If κ is regular uncountable and R is a ring of size $< \kappa$, the following are equivalent:

1. *$\mathcal{C}$ is a κ-deconstructible class of R-modules.*
2. *Whenever $R \in \mathfrak{N} \prec_1 (V, \in)$ and $\mathfrak{N} \cap \kappa$ is transitive, then*

$$\forall C \left(C \in \mathfrak{N} \cap \mathcal{C} \implies \mathfrak{N} \cap C \in \mathcal{C} \text{ and } \frac{C}{\mathfrak{N} \cap C} \in \mathcal{C} \right).$$

abbreviation:

$$\text{“}\mathbf{club}_{\boldsymbol{\kappa}} \mathfrak{N}\ \forall C \left(C \in \mathfrak{N} \cap \mathcal{C} \implies \mathfrak{N} \cap C \in \mathcal{C} \text{ and } \frac{C}{\mathfrak{N} \cap C} \in \mathcal{C} \right).\text{”}$$

("and $\mathfrak{N} \cap \kappa$ is transitive" is redundant when $\kappa = \omega_1$.)

I'll sketch the $\Uparrow$ proof. The $\Downarrow$ direction is more involved, but not used in my applications.

Outline of $\Uparrow$ direction

Assume

$$\text{club}_\kappa \mathfrak{N} \ \ \forall C \ \left(C \in \mathfrak{N} \cap \mathcal{C} \implies \mathfrak{N} \cap C \in \mathcal{C} \text{ and } \frac{C}{\mathfrak{N} \cap C} \in \mathcal{C} \right) \ \ (*)$$

Suppose $M \in \mathcal{C}$. **Want:** to write M as union of some smooth $\vec{M}$ where each $M_{\xi+1}/M_\xi$ is in $\mathcal{C}$ and of size $< \kappa$.

1. Fix a smooth $\in$ and $\subset$-increasing chain $\langle \mathfrak{N}_\zeta \ : \ \zeta < \text{cf}(|M|) \rangle$ of Σ_1-elementary submodels of $(V, \in)$ s.t. $M \subset \bigcup_\zeta \mathfrak{N}_\zeta$, $M \in \mathfrak{N}_0$, $\mathfrak{N}_\zeta \cap \kappa$ is transitive, and $|\mathfrak{N}_\zeta| < |M|$ for each $\zeta < \text{cf}(|M|)$. Then for each ζ:

$$\frac{\mathfrak{N}_{\zeta+1} \cap M}{\mathfrak{N}_\zeta \cap M} \simeq \mathfrak{N}_{\zeta+1} \cap \frac{\boldsymbol{M}}{\boldsymbol{\mathfrak{N}_\zeta \cap M}}.$$

2. Red term is in $\mathcal{C}$ by (*).
3. Then the entire RHS is in $\mathcal{C}$, again by (*) (since the red term is element of $\mathfrak{N}_{\zeta+1} \cap \mathcal{C}$).

Proof of $\Uparrow$ direction (cont.)

Consider the smooth chain

$$\left\langle \mathfrak{N}_\zeta \cap M \ : \ \zeta < \mathrm{cf}\big(|M|\big) \right\rangle \tag{2}$$

- Just saw that each $\frac{\mathfrak{N}_{\zeta+1} \cap M}{\mathfrak{N}_\zeta \cap M}$ was in $\mathcal{C}$.
- That quotient is of size $\leq |\mathfrak{N}_{\zeta+1}| < |M|$.
- So by Induction Hypothesis, each $\frac{\mathfrak{N}_{\zeta+1} \cap M}{\mathfrak{N}_\zeta \cap M}$ can be written as a union of a smooth chain

$$\left\langle \frac{X^\zeta_\xi}{\mathfrak{N}_\zeta \cap M} \ : \ \xi < \rho^\zeta \right\rangle$$

  where adjacent quotients are in $\mathcal{C}$, **and of size** $< \kappa$.
- Concatenating the $\langle X^\zeta_\xi \ : \ \xi < \rho^\zeta \rangle$ (across all $\zeta < \mathrm{cf}(|M|)$) yields the desired filtration of M. (Note $X^\zeta_{\xi+1}/X^\zeta_\xi \simeq \left(\frac{X^\zeta_{\xi+1}}{\mathfrak{N}_\zeta \cap M}\right) / \left(\frac{X^\zeta_\xi}{\mathfrak{N}_\zeta \cap M}\right)$

Outline

Vopěnka's Principle (VP)

Suppose R is a ring and P is a k-ary relation on R-modules (and is isomorphism-invariant).

Assume VP. Bagaria's characterization of VP in terms of C(n)-extendibles ([1]) yields that for any natural number n, there is $\kappa > |R|$ such that:

for any set b there is a $\mathfrak{N} \prec_n (V, \in)$ such that

- *$b \in \mathfrak{N}$, $|\mathfrak{N}| < \kappa$, and $\mathfrak{N} \cap \kappa \in \kappa$;*
- *For every $M_1, \ldots, M_k \in \mathfrak{N}$,*

$$P\big(M_1, \ldots, M_k\big) \iff P\big(\mathfrak{N} \cap M_1, \ldots, \mathfrak{N} \cap M_k\big)$$

(See Corollary A.2 of [3]).

VP (cont.)

So for any class $\mathcal{C}$ of modules, VP gives us a κ such that

$$\textbf{stat}_\kappa \mathfrak{N} \ \ \forall M \ \left(M \in \mathfrak{N} \cap \mathcal{C} \implies \mathfrak{N} \cap M \in \mathcal{C} \right). \tag{3}$$

Recall that $\mathcal{C}$ being κ-deconstructible is roughly equivalent to

$$\textbf{club}_\kappa \mathfrak{N} \ \ \forall M \ \left(M \in \mathfrak{N} \cap \mathcal{C} \implies \mathfrak{N} \cap M \in \mathcal{C} \textbf{ and } \frac{M}{\mathfrak{N} \cap M} \in \mathcal{C} \right).$$

If $\mathcal{C}$ has nice quotient behavior w.r.t. elementary submodels then we can convert the $\textbf{stat}_\kappa$ to $\textbf{club}_\kappa$, and get the $\frac{M}{\mathfrak{N} \cap M} \in \mathcal{C}$ part too. Involves very basic homological algebra. See Cox [3] (esp. Section 5) for examples.

Main applications and open problems

1. (C. [3]): VP implies deconstructibility of the class of $\mathfrak{X}$-Gorenstein Projectives (for all $\mathfrak{X}$ and over all rings)
2. (C. [4]): CON(VP) implies that Salce's Problem is independent of ZFC

Question

Is VP, or any large cardinal at all, needed for these?

[1] Joan Bagaria, Carles Casacuberta, A. R. D. Mathias, and Jiří Rosický, *Definable orthogonality classes in accessible categories are small*, J. Eur. Math. Soc. (JEMS) **17** (2015), no. 3, 549–589.

[2] L. Bican, R. El Bashir, and E. Enochs, *All modules have flat covers*, Bull. London Math. Soc. **33** (2001), no. 4, 385–390, DOI 10.1017/S0024609301008104. MR1832549

[3] Sean Cox, *Maximum deconstructibility in module categories*, Annals of Pure and Applied Algebra (to appear).

[4] ———, *Salce's problem on cotorsion pairs is undecidable*, under review, available at `https://arxiv.org/abs/2103.06687`.

[5] Paul C. Eklof and Jan Trlifaj, *How to make Ext vanish*, Bull. London Math. Soc. **33** (2001), no. 1, 41–51.