Homological algebra, elementary submodels, and stationary logic

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Two “approximation problems” from homological algebra:

Question 1 (Salce, 1970’s): Is every cotorsion pair of classes of abelian groups complete?

- Eklof-Shelah 2003: **Consistently, no**.
- Cox 2021: **Consistently, yes**

Question 2: (various, early 2000s) Does every module have a Gorenstein Projective precover?

- Šaroch, Cox, independently 2020: **Consistently, yes**

Key tool: set-theoretic *elementary submodel* arguments, particularly a new characterization of **deconstructible classes**

Will focus on the solution to Question 2

Outline

Toy argument

Quotients of free abelian groups are not always free. (E.g., $\mathbb{Z} \ / \ 2\mathbb{Z}$).

However:

Lemma (freeness "a.e." closed under subgroups and quotients)

If F is a free abelian group and $F \in \mathfrak{N} \prec_{\Sigma_1} (V, \in)$, then both $\mathfrak{N} \cap F$ and $\frac{F}{\mathfrak{N} \cap F}$ are free.

Proof: By elementarity, there is a $B \in \mathfrak{N}$ that is a free basis for F. Clearly $\mathfrak{N} \cap B$ is a free basis for $\mathfrak{N} \cap F$. But why is the quotient free?

Claim: $F = (\mathfrak{N} \cap F) \oplus \langle B \setminus \mathfrak{N} \rangle$. *Pf:* Suppose z is a nonzero element of $\mathfrak{N} \cap F$. Since $z, B \in \mathfrak{N} \prec V$, it follows that $\mathrm{sprt}_B(z) \in \mathfrak{N}$. It's a finite set, so also $\mathrm{sprt}_B(z) \subset \mathfrak{N}$. So z cannot *also* be a nontrivial linear combination of members of $B \setminus \mathfrak{N}$. $\square$

The claim implies $\frac{F}{\mathfrak{N} \cap F} \simeq \langle B \setminus \mathfrak{N} \rangle$.

Corollary: if F is free and $F \in \mathfrak{N} \prec_1 (V, \in) \ldots$

…then every morphism with domain $\mathfrak{N} \cap F$ extends to one with domain F (and the same codomain):

$$\begin{tikzcd}
F = (\mathfrak{N} \cap F) \oplus \langle B \setminus \mathfrak{N} \rangle \arrow[rr, dashed] & & M \\
\mathfrak{N} \cap F \arrow[u, hook, "\mathrm{id}"] \arrow[urr] & &
\end{tikzcd}$$

Slight elaboration yields the same when "free abelian group" is replaced by "projective module".
(Kaplansky's Theorem)

Where elementary submodels *really* help

Often encounter (long) exact sequences of projective modules:

$$\underbrace{\cdots \longrightarrow P_{-1} \xrightarrow{\pi_{-1}} P_0 \xrightarrow{\pi_0} P_1 \xrightarrow{\pi_1} \cdots}_{\boldsymbol{P_\bullet}}$$

If $P_\bullet \in \mathfrak{N} \prec_1 (V, \in)$, define

$$\underbrace{\cdots \longrightarrow \mathfrak{N} \cap P_{-1} \xrightarrow{\pi_{-1} \restriction \mathfrak{N}} \mathfrak{N} \cap P_0 \xrightarrow{\pi_0 \restriction \mathfrak{N}} \mathfrak{N} \cap P_1 \xrightarrow{\pi_1 \restriction \mathfrak{N}} \cdots}_{\boldsymbol{P_\bullet} \restriction \mathfrak{N}}$$

$P_\bullet \restriction \mathfrak{N}$ is still exact, by elementarity of $\mathfrak{N}$.

Induces a short exact sequence of complexes of projective modules

Lemma

Suppose:

1. $P_\bullet$ *is an exact complex of projective R-modules; and*
2. $R \cup \{R, P_\bullet\} \subset \mathfrak{N} \prec_1 (V, \in)$.

Then for any module M*, applying* $Hom(-, M)$ *to the short exact sequence*

$$0_\bullet \longrightarrow P_\bullet \restriction \mathfrak{N} \longrightarrow P_\bullet \longrightarrow P_\bullet/\mathfrak{N} \longrightarrow 0_\bullet$$

yields another short exact sequence of complexes:

$$0_\bullet \longleftarrow Hom\big(P_\bullet \restriction \mathfrak{N}, M\big) \longleftarrow Hom\big(P_\bullet, M\big) \longleftarrow Hom\big(P_\bullet/\mathfrak{N}, M\big) \longleftarrow 0_\bullet$$

The individual terms (complexes) in the latter can fail to be exact. But if 2 out of 3 of them are exact, then so is the 3rd ("3-by-3 lemma" from homological algebra)

Proof sketch of the lemma

$$
\begin{tikzcd}
\mathbf{0}_\bullet \arrow[d] & \dots \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \dots \\
\boldsymbol{P}_\bullet \restriction \mathfrak{N} \arrow[d] & \dots \arrow[r] & \mathfrak{N} \cap P_{-1} \arrow[r] \arrow[d] & \mathfrak{N} \cap P_0 \arrow[r] \arrow[d] & \mathfrak{N} \cap P_1 \arrow[r] \arrow[d] & \dots \\
\boldsymbol{P}_\bullet \arrow[d] & \dots \arrow[r] & P_{-1} \arrow[r] \arrow[d] & P_0 \arrow[r] \arrow[d] & P_1 \arrow[r] \arrow[d] & \dots \\
\boldsymbol{P}_\bullet/\mathfrak{N} \arrow[d] & \dots \arrow[r] & \frac{P_{-1}}{\mathfrak{N}\cap P_{-1}} \arrow[r] \arrow[d] & \frac{P_0}{\mathfrak{N}\cap P_0} \arrow[r] \arrow[d] & \frac{P_1}{\mathfrak{N}\cap P_1} \arrow[r] \arrow[d] & \dots \\
\mathbf{0}_\bullet & \dots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \dots
\end{tikzcd}
$$

Saw earlier that for any M, every $\pi : \mathfrak{N} \cap P_k \to M$ lifts to some $\widetilde{\pi} : P_k \to M$. ("Hom$(-, M)$ preserves exactness of each column.")
So Hom$(-, M)$ preserves exactness of the SES of complexes.

Outline

Classical homological algebra and projective resolutions

Fix a ring R. Given an R-module M, a **projective resolution of $\boldsymbol{M}$** is an exact sequence of homomorphisms

$$\dots \longrightarrow P_2 \xrightarrow{\pi_2} P_1 \xrightarrow{\pi_1} P_0 \xrightarrow{\pi_0} M \longrightarrow 0$$

where each P_n is a projective module.

Used to define/compute certain invariants/functors, e.g. to compute $\mathbf{Ext}(\boldsymbol{M}, -)$.

There are *many* projective resolutions of M, but the computation of $\mathrm{Ext}(M, -)$ does **NOT** depend on the choice of the projective resolution!

Homological algebra "relative to the class $\mathcal{G}$" ...

...attempts to do homological algebra, but where some class $\mathcal{G}$ is used *instead of* the class of projective modules.

E.g., one considers **$\mathcal{G}$-resolutions of M**:

$$\ldots \longrightarrow G_2 \xrightarrow{\pi_2} G_1 \xrightarrow{\pi_1} G_0 \xrightarrow{\pi_0} M \longrightarrow 0$$

where each $G_n \in \mathcal{G}$.

However, in order for these to be useful, one needs that $\mathcal{G}$ is a **precovering** ("right-approximating") class.

Precovering classes

(Auslander-Reiten-Smalø; Enochs-Xu)

A **$\mathcal{G}$-precover of the module M** is a homomorphism

$$G \xrightarrow{\pi} M$$

such that:

1. $G \in \mathcal{G}$; and
2. $\forall G' \in \mathcal{G} \;\; \forall \pi' \in \mathrm{Hom}(G', M)$: $\;\pi'$ factors through π

$$\begin{tikzcd} G \arrow[r, "\pi"] & M \\ G' \arrow[u, dotted] \arrow[ur, "\pi'"'] & \end{tikzcd}$$

$\mathcal{G}$ is a **precovering class** if every module has a $\mathcal{G}$-precover.

Case of interest: $\mathcal{G}$ = the "Gorenstein Projectives"

Question ([3], [4], [5], [6], [7], [8], [9], [10], [11], [12], [13], [14])

Is the class of Gorenstein Projective modules a precovering class (over every ring)?

Gillespie [9]: "... *the most fundamental question in Gorenstein homological algebra is whether or not Gorenstein injective preenvelopes (* **resp. Gorenstein projective precovers** *) exist*."

- **Theorem** (Šaroch): Yes, if there is a proper class of strongly compact cardinals
- I independently proved it from a proper class of supercompacts. My proof generalizes to other variants too (e.g. the Ding Projectives, the $\mathcal{X}$-Gorenstein Projectives).

Outline

Deconstructible classes

Definition (Eklof, Enochs, Trlifaj)

A class $\mathcal{C}$ of modules is **κ-deconstructible** if for every $C \in \mathcal{C}$, C can be written as the union of some $\subseteq$-increasing and $\subseteq$-continuous chain $\langle C_\xi \ : \ \xi < \zeta \rangle$ such that C_0, and each $C_{\xi+1}/C_\xi$, is:

1. in $\mathcal{C}$; and
2. is $< \kappa$-generated.

$\mathcal{C}$ is **deconstructible** if it is κ-deconstructible for some κ.

Theorem (Saorin-Stovicek, Rosicky, others?): Deconstructible* classes are precovering.

A new "top-down" characterization of deconstructibility

Theorem (C. [1])

If κ is regular uncountable and R is a ring of size $< \kappa$, the following are equivalent:

1. *$\mathcal{C}$ is a κ-deconstructible class of R-modules.*
2. *Whenever $R \in \mathfrak{N} \prec_1 (V, \in)$ and $\mathfrak{N} \cap \kappa$ is transitive, then*

$$\forall C \ \left(C \in \mathfrak{N} \cap \mathcal{C} \implies \mathfrak{N} \cap C \in \mathcal{C} \text{ and } \frac{C}{\mathfrak{N} \cap C} \in \mathcal{C} \right).$$

A cleaner (in my opinion) version of "Hill's Lemma".

Deconstructibility and Stationary Logic

So "$\mathcal{C}$ is κ-deconstructible" (roughly) means

$$\text{“}\mathbf{club}_{\boldsymbol{\kappa}}\mathfrak{N} \ \ \forall C \ \left(C \in \mathfrak{N} \cap \mathcal{C} \implies \underbrace{\mathfrak{N} \cap C \in \mathcal{C}}_{(i)} \text{ and } \underbrace{\frac{C}{\mathfrak{N} \cap C} \in \mathcal{C}}_{(ii)} \right).\text{”}$$

- Large cardinals often give **stat**-many $\mathfrak{N}$ s.t. (i) holds.
  - But how to get **club**-many? And what about (ii)?
  - Both issues taken care of if $\mathcal{C}$ is "eventually a.e. closed under quotients"

Eventual a.e. closure under quotients

Definition (C. [1])

We say $\mathcal{C}$ is **eventually a.e. closed under quotients** if for all large enough regular κ,

$$\mathbf{club}_\kappa \mathfrak{N}\ \forall C \left(\left(C \in \mathfrak{N} \cap \mathcal{C} \ \wedge \ \ \mathfrak{N} \cap C \in \mathcal{C} \right) \implies \frac{C}{\mathfrak{N} \cap C} \in \mathcal{C} \right).$$

1. The Gorenstein Projectives always have this property (in ZFC; basically the earlier lemma about complexes induced by $\mathfrak{N}$ and the 3-by-3 lemma).
2. Deconstructibility implies eventual a.e. closure under quotients (easy, using the new characterization of deconstructibility).
    1. Assuming Vopěnka's Principle, the converse holds!
    2. (In particular, when $\mathcal{C} =$ Gorenstein Projectives)

Main applications and open problems

1. (C. [1]): VP implies deconstructibility of (e.g.) the class of $\mathfrak{X}$-Gorenstein Projectives (for all $\mathfrak{X}$ and over all rings)
2. (C. [2]): CON(VP) implies that Salce's Problem is independent of ZFC

Question

Is VP, or any large cardinal at all, needed for these?

[1] Sean Cox, *Maximum deconstructibility in module categories*, Journal of Pure and Applied Algebra **226** (2022), no. 5.

[2] ______, *Salce's problem on cotorsion pairs is undecidable*, Bulletin of the London Mathematical Society (to appear).

[3] Nanqing Ding, Yuanlin Li, and Lixin Mao, *Strongly Gorenstein flat modules*, J. Aust. Math. Soc. **86** (2009), no. 3, 323–338.

[4] Ioannis Emmanouil, *Precovers and orthogonality in the stable module category*, J. Algebra **478** (2017), 174–194.

[5] Edgar E. Enochs and Overtoun M. G. Jenda, *Gorenstein injective and projective modules*, Math. Z. **220** (1995), no. 4, 611–633.

[6] ______, *Relative homological algebra*, De Gruyter Expositions in Mathematics, vol. 30, Walter de Gruyter & Co., Berlin, 2000.

[7] Sergio Estrada, Alina Iacob, and Sinem Odabasi, *Gorenstein flat and projective (pre)covers*, Publ. Math. Debrecen **91** (2017), no. 1-2, 111–121.

[8] Sergio Estrada, Alina Iacob, and Katelyn Yeomans, *Gorenstein projective precovers*, Mediterr. J. Math. **14** (2017), no. 1, Paper No. 33, 10.

[9] James Gillespie, *Gorenstein complexes and recollements from cotorsion pairs*, Adv. Math. **291** (2016), 859–911.

[10] ______, *On Ding injective, Ding projective and Ding flat modules and complexes*, Rocky Mountain J. Math. **47** (2017), no. 8, 2641–2673.

[11] Henrik Holm, *Gorenstein homological dimensions*, J. Pure Appl. Algebra **189** (2004), no. 1-3, 167–193.

[12] Alina Iacob, *Gorenstein homological algebra*, CRC Press, Boca Raton, FL, 2019. With a foreword by Sergio Estrada.

[13] Peter Jørgensen, *Existence of Gorenstein projective resolutions and Tate cohomology*, J. Eur. Math. Soc. (JEMS) **9** (2007), no. 1, 59–76.

[14] Daniel Murfet and Shokrollah Salarian, *Totally acyclic complexes over Noetherian schemes*, Adv. Math. **226** (2011), no. 2, 1096–1133.