Homological algebra, elementary submodels, and stationary logic

Sean Cox

Virginia Commonwealth University
scox9@vcu.edu
skolemhull.wordpress.com

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Two “approximation problems” from homological algebra:

Question 1 (Salce, 1970’s): Is every cotorsion pair of classes of abelian groups complete?

- Eklof-Shelah 2003: **Consistently, no**.
- Cox 2021: **Consistently, yes**

Question 2: (various, early 2000s) Does every module have a Gorenstein Projective precover?

- Šaroch, Cox, independently 2020: **Consistently, yes**
  - His proof: from proper class of strongly compacts
  - My proof: from proper class of supercompacts (but appears to generalize more easily, especially under VP)
- (open whether negative answer consistent)

Will focus on the Gorenstein result

Outline

Illustrative example of using elementary submodels

An R-module F is **free** if there is an R-linearly independent $B \subseteq F$ that generates F.

Free modules are not, in general, closed under submodules or under quotients!

- **Example of failure under quotients:** $R = \mathbb{Z}$. Both $\mathbb{Z}$ and $2\mathbb{Z}$ are free (as $\mathbb{Z}$-modules), but $\mathbb{Z}/2\mathbb{Z}$ is *not* free.
- **Example of failure under submodules:** Let $R = \mathbb{Z}/4\mathbb{Z}$, and set $F := R$. Then F is a free R-module, but its submodule $2\mathbb{Z}/4\mathbb{Z}$ is **not** free, because

$$\underbrace{(2+4\mathbb{Z})}_{\substack{\text{a nonzero scalar} \\ \text{from } R}} \quad \underbrace{(2+4\mathbb{Z})}_{\substack{\text{A nonzero} \\ \text{element of} \\ \text{the R-module } 2\mathbb{Z}/4\mathbb{Z}}} \quad = 0$$

An illustrative example (cont.)

But things "almost always" work nicely:

Lemma ("almost everywhere" closure of freeness under submodules and quotients)

If F is a free R-module and $(R, F) \in \mathfrak{N} \prec_1 (V, \in)$, then both $\langle \mathfrak{N} \cap F \rangle$ and $\frac{F}{\langle \mathfrak{N} \cap F \rangle}$ are free.

$\mathfrak{N} \prec_1 (V, \in)$ is vast overkill here, but such $\mathfrak{N}$'s will be convenient later on.

An illustrative example (cont.)

Proof of lemma: fix free basis B of F; **WLOG $\boldsymbol{B} \in \mathfrak{N}$** b/c $F, R \in \mathfrak{N} \prec_1 (V, \in)$.

$$x \in F \quad \Longrightarrow \quad x = \sum_{b \in \mathbf{sprt}_{\boldsymbol{B}}(\boldsymbol{x})} r_b^x b.$$

DEF: $X \subset F$ is "closed under B-supports" if $\mathrm{sprt}_B(x) \subset X$ for every $x \in X$. **Basic algebra exercise:** if X is closed under B-supports then both X and F/X are free.

CLAIM: $\langle \mathfrak{N} \cap F \rangle$ is closed under B-supports.

An illustrative example (cont.)

CLAIM: $\langle \mathfrak{N} \cap \boldsymbol{F} \rangle$ **is closed under** $\boldsymbol{B}$**-supports**. Let $z \in \langle \mathfrak{N} \cap F \rangle$.

- Then $z = \sum_{i=1}^{k} r_i x_i$ for some $r_i \in R$ and $x_i \in \mathfrak{N} \cap F$.
- Then

$$\operatorname{sprt}_B(z) \subseteq \bigcup_{i=1}^{k} \operatorname{sprt}_B(x_i) \tag{1}$$

- Each $\operatorname{sprt}_B(x_i) \in \mathfrak{N}$ $\big($ because $B, x_i \in \mathfrak{N} \prec_1 (V, \in)$ $\big)$
  - Hence, each $\operatorname{sprt}_B(x_i) \subset \mathfrak{N}$ because finite, even countable, elements of $\mathfrak{N}$ are always subsets of $\mathfrak{N}$.
  - Together with (1) this yields

$$\operatorname{sprt}_B(z) \subset \mathfrak{N}.$$

Similar argument yields same for "projective" modules (Kaplansky's Theorem)

Where elementary submodels *really* help

Often encounter (long) exact sequences of free modules:

$$\underbrace{\cdots \longrightarrow F_{-1} \xrightarrow{\pi_{-1}} F_0 \xrightarrow{\pi_0} F_1 \xrightarrow{\pi_1} \cdots}_{\boldsymbol{F_\bullet}}$$

If $F_\bullet \in \mathfrak{N} \prec_1 (V, \in)$, define

$$\underbrace{\cdots \longrightarrow \mathfrak{N} \cap F_{-1} \xrightarrow{\pi_{-1} \restriction \mathfrak{N}} \mathfrak{N} \cap F_0 \xrightarrow{\pi_0 \restriction \mathfrak{N}} \mathfrak{N} \cap F_1 \xrightarrow{\pi_1 \restriction \mathfrak{N}} \cdots}_{\boldsymbol{F_\bullet} \restriction \mathfrak{N}}$$

$F_\bullet \restriction \mathfrak{N}$ is still exact, by elementarity of $\mathfrak{N}$.

Induces a short exact sequence of complexes of free modules (next slide)

Short exact sequence of complexes induced by $\mathfrak{N}$ and $F_\bullet$

$$\begin{tikzcd}
\mathbf{0}_\bullet \arrow[d] & \dots \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \dots \\
F_\bullet \restriction \mathfrak{N} \arrow[d] & \dots \arrow[r] & \mathfrak{N} \cap F_{-1} \arrow[r] \arrow[d] & \mathfrak{N} \cap F_0 \arrow[r] \arrow[d] & \mathfrak{N} \cap F_1 \arrow[r] \arrow[d] & \dots \\
F_\bullet \arrow[d] & \dots \arrow[r] & F_{-1} \arrow[r] \arrow[d] & F_0 \arrow[r] \arrow[d] & F_1 \arrow[r] \arrow[d] & \dots \\
F_\bullet/\mathfrak{N} \arrow[d] & \dots \arrow[r] & \frac{F_{-1}}{\mathfrak{N} \cap F_{-1}} \arrow[r] \arrow[d] & \frac{F_0}{\mathfrak{N} \cap F_0} \arrow[r] \arrow[d] & \frac{F_1}{\mathfrak{N} \cap F_1} \arrow[r] \arrow[d] & \dots \\
\mathbf{0}_\bullet & \dots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \dots
\end{tikzcd}$$

Freeness of $\frac{F_n}{\mathfrak{N} \cap F_n}$ ensures $\mathfrak{N} \cap F_n$ is **direct summand** of F_n.
So for any M, every $\pi : \mathfrak{N} \cap F_n \to M$ lifts to some $\widetilde{\pi} : F_n \to M$.
("$\mathrm{Hom}(-, M)$ preserves exactness of each column.")
So $\mathrm{Hom}(-, M)$ preserves exactness of the (blue) SES of complexes.

Outline

Relative homological algebra (Auslander-Reiten-Smalø; Enochs-Xu)

- Classical homological algebra uses projective modules to define module and ring invariants (e.g. the Ext groups, global projective dimension, . . .)
- **Relative** homological algebra: attempts to use some class $\mathcal{G}$ instead of the projectives
  - Can provide new invariants (e.g. "$\mathcal{G}$-dimension" of a module or ring)

However, in order for these to be useful, one needs that $\mathcal{G}$ is a **precovering** ("right-approximating") class.

Case of interest: $\mathcal{G}$ = the "Gorenstein Projectives"

Question ([3], [4], [5], [6], [7], [8], [9], [10], [11], [12], [13], [14])

Is the class of Gorenstein Projective modules a precovering class (over every ring)?

Gillespie [9]: "... *the most fundamental question in Gorenstein homological algebra is whether or not ... Gorenstein projective precovers ... exist*."

- **Theorem** (Šaroch): Yes, if there is a proper class of strongly compact cardinals
- I independently proved it from a proper class of supercompacts. My proof generalizes to other variants too (e.g. the Ding Projectives, the $\mathcal{X}$-Gorenstein Projectives).

Outline

Deconstructible classes

Definition (Eklof)

A class $\mathcal{C}$ of modules is **κ-deconstructible** if for every $C \in \mathcal{C}$, C can be written as the union of some $\subseteq$-increasing and $\subseteq$-continuous chain $\langle C_\xi \ : \ \xi < \zeta \rangle$ such that C_0, and each $C_{\xi+1}/C_\xi$, is:

1. in $\mathcal{C}$; and
2. is $< \kappa$-generated.

$\mathcal{C}$ is **deconstructible** if it is κ-deconstructible for some κ.

Theorem (Saorin-Stovicek, Rosicky, others?): Deconstructible* classes are precovering.

A new "top-down" characterization of deconstructibility

Theorem (C. [1])

If κ is regular uncountable and R is a ring of size $< \kappa$, the following are equivalent:

1. *$\mathcal{C}$ is a κ-deconstructible class of R-modules.*
2. *Whenever $R \in \mathfrak{N} \prec_1 (V, \in)$ and $\mathfrak{N} \cap \kappa$ is transitive, then*

$$\forall C \ \left(C \in \mathfrak{N} \cap \mathcal{C} \implies \mathfrak{N} \cap C \in \mathcal{C} \text{ and } \frac{C}{\mathfrak{N} \cap C} \in \mathcal{C} \right).$$

A cleaner (in my opinion) version of "Hill's Lemma".

Deconstructibility and Stationary Logic

So "$\mathcal{C}$ is κ-deconstructible" (roughly) means

$$\text{“}\mathbf{club}_{\boldsymbol{\kappa}}\mathfrak{N}\ \ \forall C\ \left(C \in \mathfrak{N} \cap \mathcal{C} \implies \underbrace{\mathfrak{N} \cap C \in \mathcal{C}}_{(i)} \text{ and } \underbrace{\frac{C}{\mathfrak{N} \cap C} \in \mathcal{C}}_{(ii)} \right).\text{”}$$

- Often, if κ is a large cardinal, then we get $\mathbf{stat}_\kappa$-many $\mathfrak{N}$ s.t. (i) holds.
  - But how to get $\mathbf{club}_\kappa$-many? And what about (ii)?
  - Both issues taken care of if $\mathcal{C}$ is "eventually a.e. closed under quotients"

Eventual a.e. closure under quotients

Definition (C. [1])

We say $\mathcal{C}$ is **eventually a.e. closed under quotients** if for all large enough regular κ,

$$\mathbf{club}_\kappa \mathfrak{N}\ \forall C \left(\left(C \in \mathfrak{N} \cap \mathcal{C} \ \wedge \ \mathfrak{N} \cap C \in \mathcal{C} \right) \implies \frac{C}{\mathfrak{N} \cap C} \in \mathcal{C} \right).$$

1. The Gorenstein Projectives always have this property (in ZFC; see Section 5.1 of Cox [1]).
2. Deconstructibility implies eventual a.e. closure under quotients (easy, using the new characterization of deconstructibility).
    - Assuming Vopěnka's Principle, the converse holds!
    - (In particular, when $\mathcal{C} =$ Gorenstein Projectives)

Main applications and open problems

1. (C. [1]): VP implies deconstructibility (hence precovering) of the class of $\mathfrak{X}$-Gorenstein Projectives (for all $\mathfrak{X}$ and over all rings)
   - proper class of supercompacts suffices for "ordinary" Gorenstein and Ding projectives
2. (C. [2]): CON(VP) implies that Salce's Problem is independent of ZFC

Question

Is VP, or any large cardinal at all, needed for these?

[1] Sean Cox, *Maximum deconstructibility in module categories*, Journal of Pure and Applied Algebra **226** (2022), no. 5.

[2] ______, *Salce's problem on cotorsion pairs is undecidable*, Bull. Lond. Math. Soc. **54** (2022), no. 4, 1363–1374.

[3] Nanqing Ding, Yuanlin Li, and Lixin Mao, *Strongly Gorenstein flat modules*, J. Aust. Math. Soc. **86** (2009), no. 3, 323–338.

[4] Ioannis Emmanouil, *Precovers and orthogonality in the stable module category*, J. Algebra **478** (2017), 174–194.

[5] Edgar E. Enochs and Overtoun M. G. Jenda, *Gorenstein injective and projective modules*, Math. Z. **220** (1995), no. 4, 611–633.

[6] ______, *Relative homological algebra*, De Gruyter Expositions in Mathematics, vol. 30, Walter de Gruyter & Co., Berlin, 2000.

[7] Sergio Estrada, Alina Iacob, and Sinem Odabasi, *Gorenstein flat and projective (pre)covers*, Publ. Math. Debrecen **91** (2017), no. 1-2, 111–121.

[8] Sergio Estrada, Alina Iacob, and Katelyn Yeomans, *Gorenstein projective precovers*, Mediterr. J. Math. **14** (2017), no. 1, Paper No. 33, 10.

[9] James Gillespie, *Gorenstein complexes and recollements from cotorsion pairs*, Adv. Math. **291** (2016), 859–911.

[10] ______, *On Ding injective, Ding projective and Ding flat modules and complexes*, Rocky Mountain J. Math. **47** (2017), no. 8, 2641–2673.

[11] Henrik Holm, *Gorenstein homological dimensions*, J. Pure Appl. Algebra **189** (2004), no. 1-3, 167–193.

[12] Alina Iacob, *Gorenstein homological algebra*, CRC Press, Boca Raton, FL, 2019. With a foreword by Sergio Estrada.

[13] Peter Jørgensen, *Existence of Gorenstein projective resolutions and Tate cohomology*, J. Eur. Math. Soc. (JEMS) **9** (2007), no. 1, 59–76.

[14] Daniel Murfet and Shokrollah Salarian, *Totally acyclic complexes over Noetherian schemes*, Adv. Math. **226** (2011), no. 2, 1096–1133.