Analytic-invariant predictions and a theorem of Erdős

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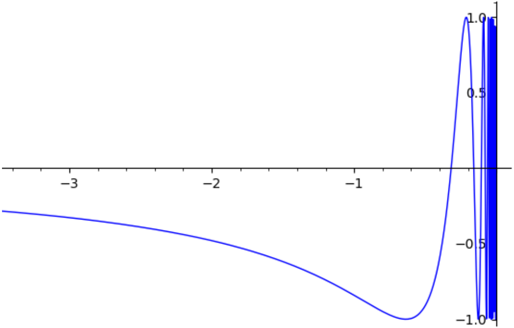
Outline

Predicting behavior of non-continuous functions

Suppose $F : \mathbb{R} \to \mathbb{R}$ is total, and $F(x) = \sin(1/x)$ when $x < 0$.
Goal: use $F|_{(-\infty,0)}$ to predict $F(0)$.



Impossible, of course. **But:**

Theorem (Hardin-Taylor [8])

Any function on $\mathbb{R}$ can be predicted almost everywhere.

arXiv 1604.00596

Getting rich quick with the Axiom of Choice

Vladimir Vovk

March 22, 2017

Abstract

This paper proposes new get-rich-quick schemes that involve trading in a financial security with a non-degenerate price path. For simplicity the interest rate is assumed zero. If the price path is assumed continuous, the trader can become infinitely rich immediately after it becomes non-constant (if it ever does). If it is assumed positive, he can become infinitely rich immediately after reaching a point in time such that the variation of the log price is infinite in any right neighbourhood of that point (whereas reaching a point in time such that the variation of the log price is infinite in any left neighbourhood of that point is not sufficient). The practical value of these schemes is tempered by their use of the Axiom of Choice.

The Hardin-Taylor theorem

Theorem (Hardin-Taylor [8])

Any function on $\mathbb{R}$ can be predicted almost everywhere.

Fix any set S (e.g., $S = \mathbb{R}$).
$\overset{\mathbb{R}}{\smile}\boldsymbol{S}$:= all functions of the form $f : (-\infty, t_f) \to S$ for some $t_f \in \mathbb{R}$

They use a wellorder of ${}^{\mathbb{R}}S$ to build a **good $\boldsymbol{S}$-predictor**: a function $\mathcal{P} : \overset{\mathbb{R}}{\smile}S \to S$ s.t. for every total $F : \mathbb{R} \to S$:

$$\mathrm{Fail}_{\mathcal{P},F} := \left\{ x \in \mathbb{R} \ : \ \mathcal{P}\left(F|_{(-\infty,x)} \right) \neq F(x) \right\}$$

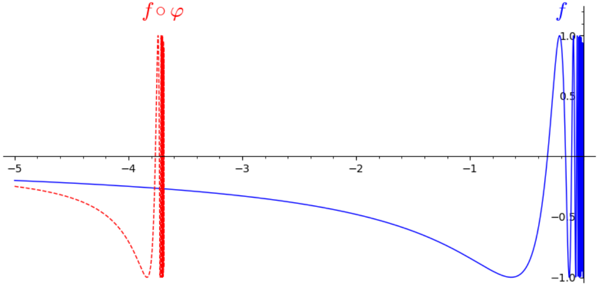
is countable.

Invariance ("anonymity") under time distortions

Fix $U \subseteq \mathrm{Homeo}^+(\mathbb{R})$. Let's say $\mathcal{P} : {}^{\overset{\mathbb{R}}{\smile}}S \to S$ is ***U*-invariant** if

$$\mathcal{P}(f) = \mathcal{P}\big(f \circ \varphi\big)$$

for all $f \in {}^{\overset{\mathbb{R}}{\smile}}S$ and all $\varphi \in U$.
E.g., say $\varphi(x) = x + e^{x+5}$ ($\in \mathrm{Homeo}^+(\mathbb{R})$)



Question (Hardin-Taylor, Bajpai-Velleman)

How invariant can good predictors be?

Invariance under time distortions (cont.)

Theorem (Bajpai-Velleman [1]):

- (positive result): There is a good *affine*-invariant predictor (for every S)
- (negative result): There is **no** good C^∞-invariant predictor (whenever $|S| = |\mathbb{R}|$)

(improved by Cox-Elpers [4])

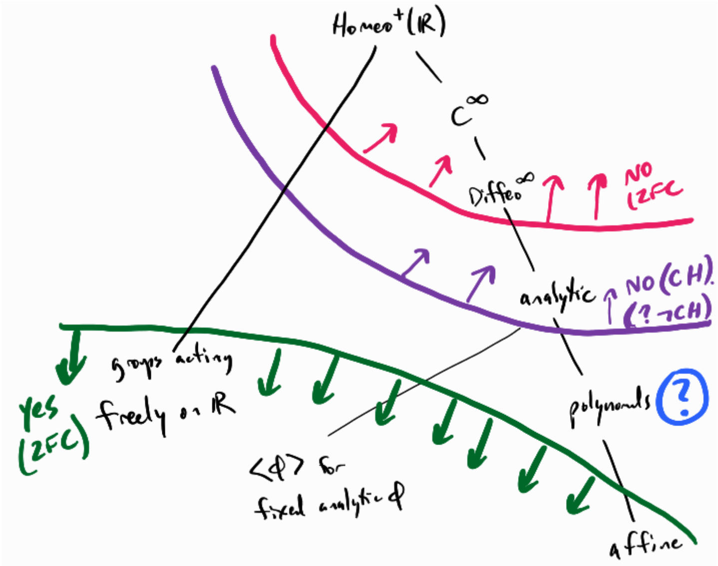
Question (Bajpai-Velleman [1])

What about (real) **analytic***-invariant predictors?*

Theorem (Cody-Cox-Lee [3])

Continuum Hypothesis implies there is **no** *good analytic-invariant predictor (whenever $|S| = |\mathbb{R}|$).*

How invariant can good predictors be? ([1], [3], [4])



Outline

1. Using AC to predict non-continuous functions
2. Sparse analytic systems and a theorem of Erdős
3. Open questions

Wetzel's Problem ($\sim$1960)

Suppose $\mathcal{F}$ is a family of analytic functions, and

$$\forall x \;\; \{f(x) \; : \; f \in \mathcal{F}\} \text{ is countable.}$$

Must $\mathcal{F}$ be countable?

Theorem (Erdős [5] 1963)

CH is equivalent to a negative answer to Wetzel's Problem.

Final line of Erdős' paper: "Paul Cohen's recent proof of the independence of the continuum hypothesis gives this problem some added interest."
Wetzel: "Once again a natural analysis question has grown horns!" ([7])

Sparse analytic systems [3]

Def: A **sparse analytic system** is a collection $\langle f_P \;:\; P \in \mathbb{R}^2 \rangle$ such that:

1. Each f_P is an analytic member of $\mathrm{Homeo}^+(\mathbb{R})$ passing through the point $P = (x_P, y_P)$;
2. For every $z \in \mathbb{R}$:

$$\left\{ f_P(z) \;:\; z \neq x_P \right\} \text{ and } \left\{ f_P^{-1}(z) \;:\; z \neq y_P \right\}$$

   are both countable.

Theorem (Cody-Cox-Lee [3])

1. *CH is equivalent to the existence of a sparse analytic system.*
2. *And existence of a sparse analytic system implies there is no good analytic-invariant predictor (whenever $|S| = |\mathbb{R}|$)*

Sparse analytic systems **block** analytic-invariant predictors

If $\langle f_P \ : \ P \in \mathbb{R}^2 \rangle$ is a sparse analytic system, let $\sim$ be the equivalence relation on $\mathbb{R}$ generated by

$$\left\{(u,v) \in \mathbb{R}^2 \ : \ (\exists P)(u \neq x_P \ \wedge \ v \neq y_P \ \wedge \ f_P(u) = v)\right\}.$$

1. Each $\sim$-equivalence class is **countable** (by sparseness of the f_P's)
2. Since f_P's are analytic and $u \sim f_P(u)$ whenever $u \neq x_P$, routine to check that any analytic-invariant attempt to predict $x \mapsto [x]_\sim$ based on past behavior must be constant (i.e., picks out a single equivalence class). So by 1, it can only be correct for countably many $x \in \mathbb{R}$.

(Nothing special about "analytic" here)

(Why must any analytic-invariant $\mathcal{P}$ constantly predict the map $E : \mathbb{R} \to \mathbb{R}/\sim$, $x \mapsto [x]_\sim$?
Suppose $x, y \in \mathbb{R}$; by analytic-invariance of $\mathcal{P}$,

$$\mathcal{P}\Big(E|_{(-\infty,y)}\Big) = \mathcal{P}\Big(E|_{(-\infty,y)} \circ f_{(x,y)}\Big).$$

And by definition of $\sim$,

$$E|_{(-\infty,y)} \circ f_{(x,y)} = E|_{(-\infty,x)}$$

So

$$\mathcal{P}\Big(E|_{(-\infty,y)}\Big) = \mathcal{P}\Big(E|_{(-\infty,x)}\Big)$$

A result in complex analysis

Theorem ((ZFC)(Cody-Lee, proving a conjecture of Cox))

Suppose $\mathcal{D}$ is a partition of $\mathbb{R}$ into countable dense subsets of $\mathbb{R}$; for $w \in \mathbb{R}$, D_w will denote the unique $D \in \mathcal{D}$ such that $w \in D$.

If W is a **countable** *set of reals and $P = (x_P, y_P) \in \mathbb{R}^2$, then there is an analytic $f : \mathbb{C} \to \mathbb{C}$ s.t. f passes through P, $f \restriction \mathbb{R}$ is increasing bijection $\mathbb{R} \to \mathbb{R}$, and for each $w \in W$:*

- $w \neq x_P \implies f(w) \in D_w$.
- $w \neq y_P \implies f^{-1}(w) \in D_w$.

(Extends Franklin [6], Barth-Schneider [2], Maurer [9], Nienhuys-Thiemann [10], Sato-Rankin [11], and others)

CH yields sparse analytic systems (inspired by Erdős [5])

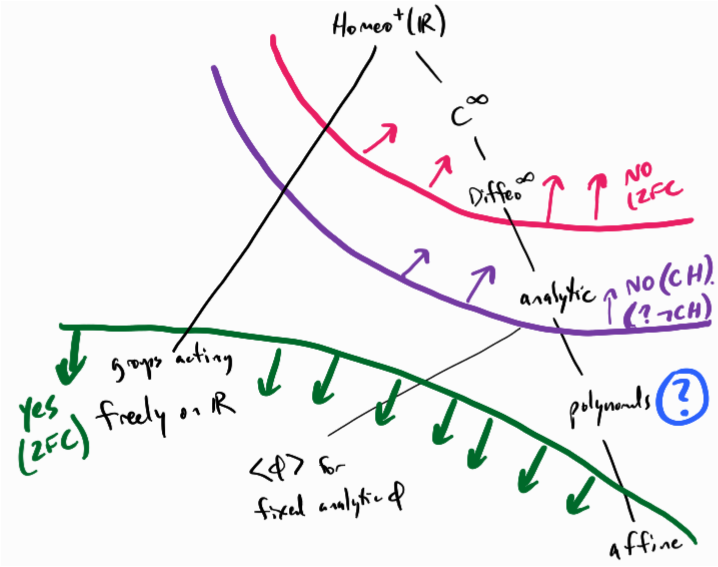
Fix any partition $\mathcal{D}$ of $\mathbb{R}$ into countable dense subsets of $\mathbb{R}$.
Assuming CH, fix enumerations $\langle w_\alpha \ : \ \alpha < \omega_1 \rangle$ of $\mathbb{R}$ and $\langle P_\alpha = (x_\alpha, y_\alpha) \ : \ \alpha < \omega_1 \rangle$ of $\mathbb{R}^2$.
For $\alpha < \omega_1$ set $W_{<\alpha} := \{w_\xi \ : \ \xi < \alpha\}$. By Cody-Lee theorem, there is analytic f_α passing through P_α s.t. for $w \in W_{<\alpha}$:
$w \neq x_\alpha \implies f_\alpha(w) \in D_w$ and $w \neq y_\alpha \implies f_\alpha^{-1}(w) \in D_w$.
Then: for any $w = w_\xi \in \mathbb{R}$,

$$\{f_\alpha(w_\xi) : w_\xi \neq x_\alpha\} \subseteq \underbrace{\{f_\alpha(w_\xi) : (\xi < \alpha) \wedge (w_\xi \neq x_\alpha)\}}_{\subseteq D_{w_\xi}\text{, which is ctble}} \cup \underbrace{\{f_\alpha(w_\xi) : \alpha \leq \xi\}}_{\text{ctble}}$$

(and similarly for the $f_\alpha^{-1}(w)$'s).
So $\langle f_\alpha \ : \ \alpha < \omega_1 \rangle$ is a sparse analytic system.

Outline

Current knowledge ([1], [3], [4])



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