How robustly can you predict the future?

Sean Cox

Virginia Commonwealth University
scox9@vcu.edu
skolemhull.wordpress.com
Partially supported by NSF grant DMS-2154141 and VCU Seed Grant

Charles U. Algebra Seminar
March 20, 2023

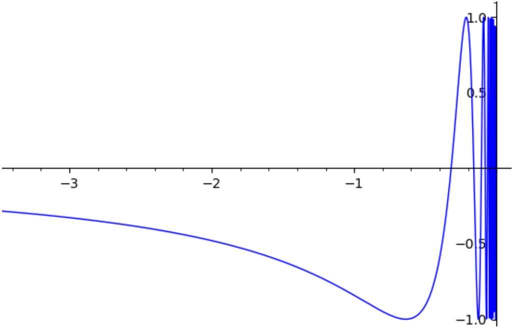
Outline

1. Predicting non-continuous functions
2. Additional requirement: invariance of predictions under time distortions
3. Lots of open problems

Can we use $F \restriction_{(-\infty,t)}$ to predict $F(t)$?

Easy, if F is known to be continuous at t; just use $\lim_{s \nearrow t} F(s)$.

But generally not possible; how would you predict $F(0)$ if $F \restriction_{(-\infty,0)}$ looks like:



Predicting non-continuous functions

Notation: $F_{<t}$ will abbreviate $F \restriction_{(-\infty,t)}$

Theorem (Hardin-Taylor [4])

There is a mapping $\mathcal{P}$ from partial functions into $\mathbb{R}$ such that for every total $F : \mathbb{R} \to \mathbb{R}$,

$$\mathcal{P}\left(F_{<t}\right) = F(t)$$

for **almost** *every $t \in \mathbb{R}$.*

(the measure zero set of "bad" t is allowed to depend on F)

arXiv 1604.00596

Getting rich quick with the Axiom of Choice

Vladimir Vovk

March 22, 2017

Abstract

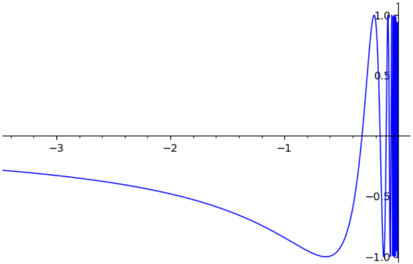
This paper proposes new get-rich-quick schemes that involve trading in a financial security with a non-degenerate price path. For simplicity the interest rate is assumed zero. If the price path is assumed continuous, the trader can become infinitely rich immediately after it becomes non-constant (if it ever does). If it is assumed positive, he can become infinitely rich immediately after reaching a point in time such that the variation of the log price is infinite in any right neighbourhood of that point (whereas reaching a point in time such that the variation of the log price is infinite in any left neighbourhood of that point is not sufficient). The practical value of these schemes is tempered by their use of the Axiom of Choice.

The version of this paper at `http://probabilityandfinance.com` (Working Paper 43) is updated most often. The journal version is to appear in *Finance and Stochastics* under the title "The role of measurability in game-theoretic probability".

The Hardin-Taylor theorem

${}^{\overset{\mathbb{R}}{\smile}}\mathbb{R}$:= real-valued functions f on domains of the form $(-\infty, t_f)$
Example: $f : (-\infty, 0) \to \mathbb{R}$, $f(t) = \sin(1/t)$. $(t_f = 0)$
A **predictor** is any function $\mathcal{P}$ with domain ${}^{\overset{\mathbb{R}}{\smile}}\mathbb{R}$ and codomain $\mathbb{R}$.

$\mathcal{P}(f)$ can be viewed as an attempt to predict what f "should/would/will" do at t_f (if t_f were in its domain):



The Hardin-Taylor Theorem (cont.)

$\mathcal{P}$ is **good** if for every (total) $F : \mathbb{R} \to \mathbb{R}$,

$$\mathrm{Fail}_{\mathcal{P},F} := \left\{ t \in \mathbb{R} \ : \ \mathcal{P}\Big(F_{<t}\Big) \neq F(t) \right\}$$

has Lebesgue measure zero.

Theorem (Hardin-Taylor)

There is a good predictor!

Uses Axiom of Choice (a wellorder of ${}^{\mathbb{R}}\mathbb{R}$)

In fact, can arrange $\mathrm{Fail}_{\mathcal{P},F}$ is always a “well-ordered” subset of $(\mathbb{R}, <)$, hence countable.

Proof of Hardin-Taylor:

1. Using Axiom of Choice, there exists a well-order $\lhd$ of all total functions from $\mathbb{R} \to \mathbb{R}$.
2. Given $f : (-\infty, t_f) \to \mathbb{R}$, let $G_f : \mathbb{R} \to \mathbb{R}$ be the $\lhd$-**least** total function that extends f, and set $\mathcal{P}(f) := G_f(t_f)$.
3. Given any total $F : \mathbb{R} \to \mathbb{R}$, consider the set

$$\mathrm{Fail}_{\mathcal{P},F} := \left\{ t \in \mathbb{R} \ : \ \underbrace{\mathcal{P}\Big(F_{<t}\Big)}_{G_{(F_{<t})}(t)} \neq F(t) \right\}$$

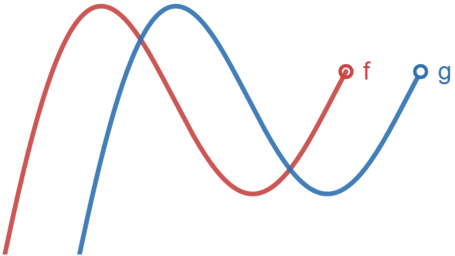
4. **Key obs:** if $a < b$ are both in $\mathrm{Fail}_{\mathcal{P},F}$, then $G_{(F_{<a})} \lhd G_{(F_{<b})}$ (hence, $\mathrm{Fail}_{\mathcal{P},F}$ is a well-ordered, hence countable, subset of $(\mathbb{R}, <)$)
    - "$\unlhd$": because of $\lhd$-minimality of $G_{(F_{<a})}$ and the fact that $F|_{(-\infty,a)} \subset F_{(-\infty,b)} \subset G_{(F_{<b})}$.
    - "$\neq$": $G_{(F_{<a})}(a) \neq F(a)$ (b/c $a \in \mathrm{Fail}_{\mathcal{P},F}$), but $G_{(F_{<b})}(a) = F_{<b}(a) = F(a)$ (because $F_{<b} \subset G_{(F_{<b})}$).

Outline

1. Predicting non-continuous functions
2. Additional requirement: invariance of predictions under time distortions
3. Lots of open problems

(horizontal) shift invariance

With more work, Hardin-Taylor also arranged for $\mathcal{P}$ to be *horizontal shift invariant*, e.g. $\mathcal{P}(f) = \mathcal{P}(g)$ in situations like this:



Question (Hardin-Taylor)

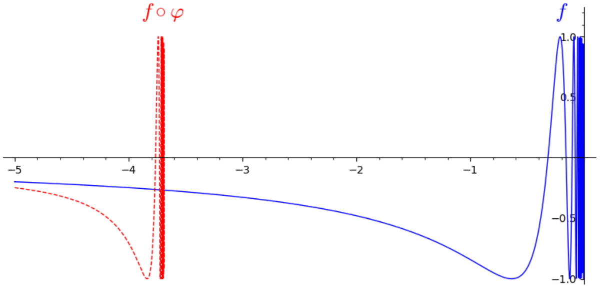
How much more invariance can we arrange?

Invariance of predictors

Fix $U \leq \left(\mathrm{Homeo}^{+}(\mathbb{R}), \circ\right)$. $\mathcal{P} : \overset{\mathbb{R}}{\smile}\mathbb{R} \to \mathbb{R}$ is ***U*-invariant** if

$$\mathcal{P}(f) = \mathcal{P}\left(f \circ \varphi\right)$$

for all $f \in \overset{\mathbb{R}}{\smile}\mathbb{R}$ and all $\varphi \in U$.
E.g., if $\varphi(t) = t + e^{t+5}$ and $\varphi \in U$, then U-invariance requires $\mathcal{P}(f \circ \varphi) = \mathcal{P}(f)$:



How invariant can good predictors be?

Precise version of their question:

Question (Hardin-Taylor [1])

For which $U \subseteq Homeo^{+}(\mathbb{R})$ do there exist good, U-invariant predictors?

Positive results

Theorem (Hardin-Taylor [5]): there is a good translation-invariant predictor.
Theorem (Bajpai-Velleman [1]): there is a good affine-invariant predictor.
Theorem (Cox-Elpers [3]): there is a good U-invariant predictor, whenever $U \leq \text{Homeo}^{+}(\mathbb{R})$ has the following properties:

1. The commutator subgroup of U acts freely on $\mathbb{R}$; and
2. non-identity members of U have at most one fixed point.

Key ingredient: Hölder's Theorem that Archimedean groups are abelian

Well-known consequence of Hölder's Theorem

Suppose $C \leq \mathrm{Homeo}^+(\mathbb{R})$ and C acts freely on $\mathbb{R}$ (i.e. if $\mathrm{id} \neq \varphi \in C$ then φ has no fixed points).

Using Intermediate Value Theorem and the assumption about C, can show

$$\varphi < \psi \iff_{\text{def}} \forall t \; \varphi(t) < \psi(t)$$

is a total, Archimedean order on C. So by Hölder's Theorem, C is abelian.

(Makes it easier to arrange C-invariance of functions)

Negative results

But there are limits, even with Axiom of Choice at our disposal!

Theorem (Bajpai-Velleman [1])

There is **NO** *good C^∞-invariant predictor.*

Theorem (Cox-Elpers [3])

There is **NO** *good D^∞-invariant predictor.*

What about subgroups of $\mathrm{Homeo}^+(\mathbb{R})$ ...

... between affine and C^∞?

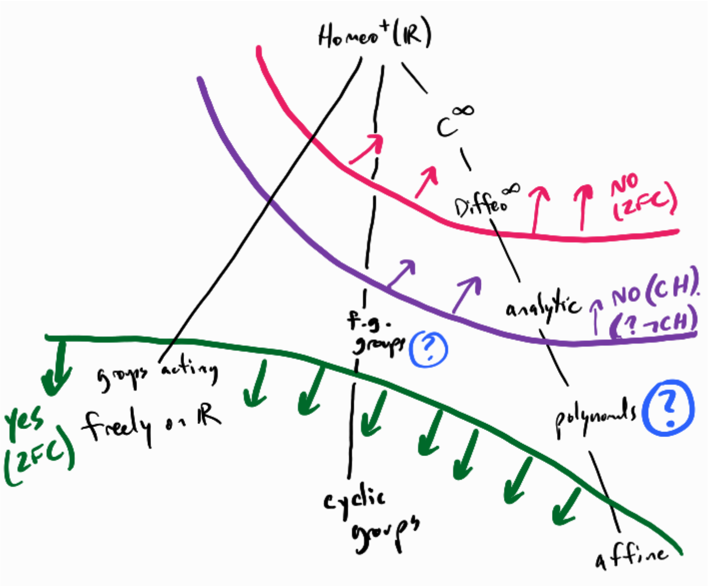
Question (Bajpai-Velleman [1])

Do there exist (real) **analytic***-invariant good predictors?*

Theorem (Cody-Cox-Lee [2])

Assuming the Continuum Hypothesis (CH): **no***.*

How invariant can good predictors be? ([1], [2], [3])



Very accessible literature

Uses material from only first half of typical undergrad analysis and algebra courses:

- Hardin-Taylor: *A peculiar connection between the axiom of choice and predicting the future* (Amer. Math. Monthly)
- Bajpai-Velleman: *Anonymity in predicting the future* (Amer. Math. Monthly)
- Cox-Elpers: *How robustly can you predict the future?* (Canadian J. Math)

Uses a tiny bit of set theory (countable ordinals):

- Cody-Cox-Lee: *Sparse Analytic Systems* (arxiv)

Outline

Are there good U-invariant predictors when

$U \leq \text{Homeo}^+(\mathbb{R})$ and:

1. U=analytic?
   - If CH holds, no (Cody-Cox-Lee). But what if CH fails?
2. U is generated by 2 functions?
   - Yes if U is *cyclic* (Cox-Elpers showed this when U is generated by an analytic function, but the argument works for any cyclic subgroup of $\text{Homeo}^+(\mathbb{R})$)
3. $U = \langle\text{strictly increasing polynomials}\rangle$?
4. All known examples of U for which good U-invariant predictors exist are **metabelian** (cummutator subgroup is abelian). Are there non-metabelian examples?

[1] Dvij Bajpai and Daniel J. Velleman, *Anonymity in predicting the future*, Amer. Math. Monthly **123** (2016), no. 8, 777–788.

[2] Brent Cody, Sean Cox, and Kayla Lee, *Sparse analytic systems*, submitted, available at `https://arxiv.org/abs/2208.13725`.

[3] Sean Cox and Matthew Elpers, *How robustly can you predict the future?*, Canadian Journal of Mathematics (to appear), available at `https://arxiv.org/abs/2208.09722`.

[4] Christopher S. Hardin and Alan D. Taylor, *A peculiar connection between the axiom of choice and predicting the future*, Amer. Math. Monthly **115** (2008), no. 2, 91–96.

[5] ———, *The mathematics of coordinated inference*, Developments in Mathematics, vol. 33, Springer, Cham, 2013.